Spring 2017 McNabb GDCTM Contest
Pre-Algebra

NO Calculators Allowed

1. Hezy paid \$19.04 for a shirt that was marked 15% off. What was the original price of the shirt?
2. Which is larger: 7^{14} or 14^7?
3. What is the positive difference between the sum of the first two-hundred positive multiples of four and the first one-hundred positive multiples of eight?
4. Find the smallest positive integer n so that $28n$ is a perfect square.
5. Find the units digit of 17^{2017}.
6. Let $x = 0.\overline{57}$ and $y = 0.\overline{03}$. Write $x + y$ in simplified fraction form.
7. Find the positive integer n so that $n/17$ gives the best possible approximation to $\sqrt{2}$.
8. Find the sum of the eight largest four digit pallindromes.
9. What is the remainder when 123456654321 is divided by seven?
10. A basketball league has two divisions of nine teams each. In a season, each team plays every other team in their division twice, and every team in the other division once. How many games are played in total in one season?
11. Find the 2017th decimal place in the decimal expansion of the fraction 1/37.
12. In how many ways can two subsets of

    $$S = \{a, b, c, d, e, f\}$$

    be chosen so that their union is S and their intersection contains three letters? The order of the subsets is not material.
13. Simplify $\sqrt[3]{970299}$.
14. Let a, b, c, and d, be four distinct integers whose product is 1024. Find the least possible value of their sum.
15. In how many ways can $1,000,000$ be written as the product of three positive integers, where the order of the factors matters?

Spring 2017 McNabb GDCTM Contest
Algebra One

NO Calculators Allowed

1. Hezy paid \$19.04 for a shirt that was marked 15% off. What was the original price of the shirt?

2. Let $x = 0.\overline{57}$ and $y = 0.\overline{03}$. Write $x + y$ in simplified fraction form.

3. Define the function
$$g(a, b, c) = \left(\frac{a^3 + b^3 + c^3}{a + b + c}\right)^2$$
Find the value of $g(3, 4, 5)$.

4. Let x, y, and z be the solutions of the system
$$\begin{aligned} x + 2y - z &= -1 \\ 2x - y + z &= 9 \\ x + 3y + 3z &= 6 \end{aligned}$$
Find the value of $10x + 2y + z$.

5. A circular pond with volume 36π cubic feet and depth 4 feet is having a circular walkway built around it. The walkway should be 4 feet wide and be sunk 2 feet into the ground. What volume of concrete is needed to build the walkway? Answer in cubic feet.

6. True or False:
$$\frac{\sqrt{5} + \sqrt{7}}{2} > \sqrt{6}$$

7. Billy sells \$3471 worth of chocolate boxes. He sells two kinds of boxes, a milk chocolate mix at \$15 per box, and a dark chocolate assortment at \$14 per box. If he sells a total of 236 boxes, how many boxes of the dark chocolate assortment does he sell?

8. How many positive integers less than five-hundred are relatively prime to either 16 or 27 but not both?

9. Find distinct positive integers (p, q, r) so that $p < q < r$ and
$$\frac{7}{11} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$$
Write your answer as (p, q, r)

10. Eve is standing at the exact center of an orchard, whose trees are evenly spaced in 16 rows, with 16 trees in each row, making a square array. How many of the trees that form the boundary of the orchard can she see? Treat the trees as skinny poles.

11. Three consecutive integers have the property that the cube of their sum minus nine times the sum of their cubes equals -918. What is the smallest of these integers?

12. A quadratic function p satisfies $p(7) = -5$, $p(8) = 4$, and $p(9) = -7$. Find the value of $p(6)$.

13. Store-owner Josie wants to price items in dollars and cents at her store between \$10.00 and \$20.00 inclusive so that even after a 15% discount they are still exact. For instance, she would not want to price an item at \$11.25 because after a 15% discount, the price would be \$9.5625, which is not of the form $\$ab.cd$. How many initial prices meet Josie's requirements?

14. Let a, b, c, and d be real numbers satisfying

$$\begin{aligned} a^2 + b^2 &= 37 \\ c^2 + d^2 &= 26 \\ ac - bd &= 11 \end{aligned}$$

Find the largest possible value of $ad + bc$.

15. Napoleon's army is on retreat from Russia back to France forming a column 20 miles long. Napoleon on his horse at the back of the column proceeds to the front of the column, then returns to the back, taking 7 hours altogether. If his army marches at 3 mph, how fast does Napoleon's horse trot?

Spring 2017 McNabb GDCTM Contest
Geometry

NO Calculators Allowed

1. Find the volume of a sphere of radius 3.

2. In triangle ABC with $\angle C$ right, a square built on side AC has area 64 while a square built on side AB has area 100. Find the length of side BC.

3. Billy sells \$3471 worth of chocolate boxes. He sells two kinds of boxes, a milk chocolate mix at \$15 per box, and a dark chocolate assortment at \$14 per box. If he sells a total of 236 boxes, how many boxes of the dark chocolate assortment does he sell?

4. Let
$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h} = \frac{4}{5}$$
then what is the value of
$$\frac{a^2 + c^2 + e^2 + g^2}{b^2 + d^2 + f^2 + h^2}$$
?

5. A circular pond with volume 36π cubic feet and depth 4 feet is having a circular walkway built around it. The walkway should be 4 feet wide and be sunk 2 feet into the ground. What volume of concrete is needed to build the walkway? Answer in cubic feet.

6. Given rectangle $ABCD$ let E and F be the midpoints of sides AB and CD respectively. Draw a circle with diameter EF of length 10. If the ratio of the area of the circle to the area of the rectangle is $\pi/8$, find the perimeter of the rectangle.

7. A quarter-circle is inscribed in a $30^\circ - 60^\circ - 90^\circ$ triangle, with its center at the vertex of the right angle. If the area of the quarter circle is 36π, what is the area of the triangle?

8. Convex hexagon $ABCDEF$ has
$$EF = FA = AB = BC = 2$$
$$\angle F = \angle A = \angle B = 150^\circ$$
$$\angle E = \angle D = \angle C = 90^\circ$$
What is the area of this hexagon?

9. In $\triangle ABC$, $AB = 16$, $BC = 5$, and $\angle B = 120$ degrees. Find AC.

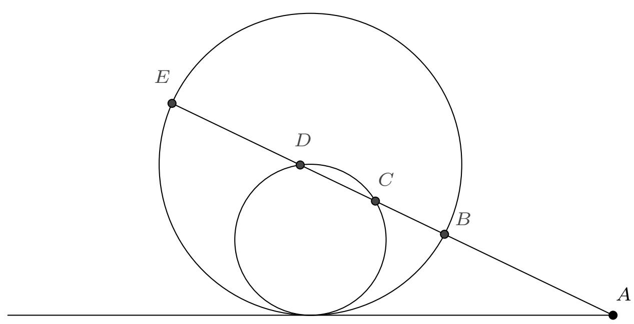
10. Let (a, b) be fixed positive real numbers. Find the area of the parallelogram formed by the four lines
$$y = ax - b$$
$$y = ax + b$$
$$y = -ax - b$$
$$y = -ax + b$$

in terms of a and b.

11. A right triangle has all integer side lengths, the smallest of which is eleven. Find the area of the triangle.

12. Rectangle $ABCD$ is a rectangular billiard table with $AD = 1$ and $AB = 3$. A ball is at point P on side CD with $DP = 1$. A player aims the ball at point Q on side BC so that after three carooms (bounces) the ball will be headed back to where it started. Find CQ.

13. Circle R and smaller circle S are internally tangent to each other at point P and both externally tangent to line n, also at point P. A second line m cuts circle R at points B and E, circle S at points C and D, and line n at point A, so that points A, B, C, D, E occur in that order on m. If $AB = 3$, and $BC = CD = 1$, find DE.



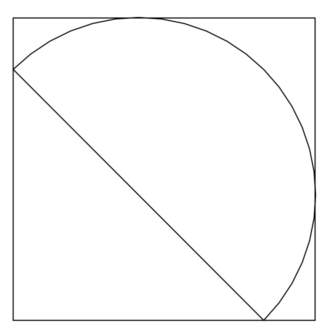
14. A circle of area 36π is inscribed in square $ABCD$. Side AB is extended past B to point E and side AD is extended past D to point F in such a way that C lies on EF. Find the minimum possible area of $\triangle EAF$.

15. In rectangle $ABCD$, let F be the midpoint of AB, and points E and G be the midpoints of AF and FB respectively. Draw diagonal AC and segments DE, DF and DG intersecting AC at points H, I, J respectively. Find the ratio HI/IJ.

Spring 2017 McNabb GDCTM Contest
Algebra Two

NO Calculators Allowed

1. Which is larger: 7^{14} or 14^7?
2. Find just one ordered pair of integers (m, n) so that
$$m^2 + mn + n^2 = 39$$
3. Write 621 as the sum of three perfect squares.
4. Let (a, b) be fixed positive real numbers. Find the area of the parallelogram formed by the four lines
$$y = ax - b$$
$$y = ax + b$$
$$y = -ax - b$$
$$y = -ax + b$$
in terms of a and b.
5. Find the minimum possible value of $ax + by + cz$ if $\{a, b, c\} = \{4, 5, 6\}$ and $\{x, y, z\} = \{7, 8, 9\}$.
6. A semicircle is inscribed in a square as shown. This means that both endpoints of the diameter of the semicircle lie on the square, and at the other two points of contact, the side of the square is tangent to the semicircle. Find the ratio of the area of the semicircle to the area of the square.



7. Let x, y, and z be the solutions of the system

$$\begin{aligned} x + 2y - z &= -1 \\ 2x - y + z &= 9 \\ x + 3y + 3z &= 6 \end{aligned}$$

Find the value of $10x + 2y + z$.

8. Four cards are randomly removed from a standard deck of 52 playing cards. Then one of the four cards that were removed is chosen at random. What is the probability that card is an ace?

9. In how many ways can two subsets of

$$S = \{a, b, c, d, e, f\}$$

be chosen so that their union is S and their intersection contains three letters? The order of the subsets is not material.

10. Find the coefficient of x^6 when

$$(x^3 - 6x^2 + 11x - 6)^3$$

is expanded and simplified.

11. Let $\angle A = 60°$. A circle of radius 1 is dropped into this angle so it fits as snugly as possible (closest to point A). What is the radius of the circle that will fit snugly on top of the first circle and be tangent to both sides of the angle?

12. In a city of $100,000$ souls, 400 carry gene G. A test for gene G is 99% accurate in both directions, meaning that if a person has gene G, the test will be positive 99% of the time and that if a person does not have the gene G the test will be positive 1% of the time. A randomly selected resident of the city tests positive for gene G. What is the probability that this person actually has gene G? Answer as a fraction. Warning: you may be surprised by the answer!

13. Let r, s, and t be the roots of the polynomial $x^3 - 4x^2 - 11x + 30$. Find the value of $r^2 + s^2 + t^2$.

14. It is known that every isosceles trapezoid can be inscribed in a circle. Find the radius of that circle if the side lengths of the isosceles trapezoid are 3, 5, 9, and 5.

15. Let a, b, c, and d be real numbers satisfying

$$\begin{aligned} a^2 + b^2 &= 37 \\ c^2 + d^2 &= 26 \\ ac - bd &= 11 \end{aligned}$$

Find the largest possible value of $ad + bc$.

Spring 2017 McNabb GDCTM Contest
PreCalculus

NO Calculators Allowed

1. True or False:
$$\frac{\sqrt{5}+\sqrt{7}}{2} > \sqrt{6}$$

2. For what real value(s) of the constant k does the equation
$$\frac{x}{x+3} - \frac{1}{x-3} = \frac{kx}{x^2-9}$$
have exactly one real valued solution?

3. What is the remainder when 123456654321 is divided by seven?

4. Simplify $\sqrt[3]{970299}$.

5. Find one ordered pair of positive integers (m, n) so that
$$21m - 34n = 1$$

6. Write $\cos 15°$ in simplified radical form.

7. Find one complex number z in the form $a + bi$ where a and b are real with $a \neq 0$ and $b \neq 0$, such that
$$z^6 + 6z^5 + 15z^4 + 20z^3 + 15z^2 + 6z = 0$$

8. In a city of $100,000$ souls, 400 carry gene G. A test for gene G is 99% accurate in both directions, meaning that if a person has gene G, the test will be positive 99% of the time and that if a person does not have the gene G the test will be positive 1% of the time. A randomly selected resident of the city tests positive for gene G. What is the probability that this person actually has gene G? Answer as a fraction. Warning: you may be surprised by the answer!

9. Solve for x:
$$\begin{vmatrix} x^2 & x & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{vmatrix} = 0$$

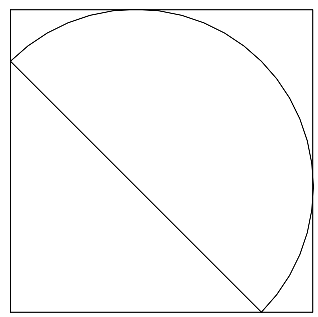
10. Find the number of ways to mail ten distinct books if two packages contain three books each and one package contains four books?

11. A coach takes orders for seven of his players. Each player chooses exactly one of these items: hot dogs, nachos, chicken tenders, salad. How may different *orders* can occur? That is, do not take into account which specific player gets a given item. (When the coach arrives each player is happy!)

12. Starting at the origin take unit steps to the right or left with equal probability. Stop when you reach the points -3 or 3. What is the average number of steps you will take, when you repeat this experiment over and over?

13. A regular hexagon $ABCDEF$ has center G and side lengths 2. The hexagon is divided into six equilateral triangles in the usual way (such as $\triangle GAB$, etc...). The centers of mass of these six equilateral triangles form hexagon $HIJKLM$. Find the ratio of the area of $HIJKLM$ to the area of $ABCDEF$.

14. A semicircle is inscribed in a square as shown. This means that both endpoints of the diameter of the semicircle lie on the square, and at the other two points of contact, the side of the square is tangent to the semicircle. Find the ratio of the area of the semicircle to the area of the square.



15. Six children are on a merry-go-round with six seats, all facing the same way. They decide to change how they are seated so that each child will have a different child in front of them (all still facing the same way). In how many ways can this be done?

Spring 2017 McNabb GDCTM Contest
Calculus

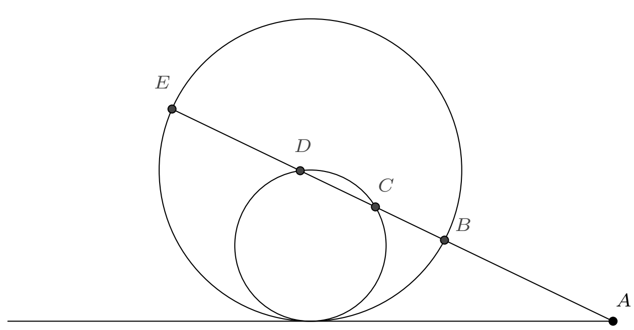
NO Calculators Allowed

Assume all variables are real unless otherwise stated in the problem.

1. Four cards are randomly removed from a standard deck of 52 playing cards. Then one of the four cards that were removed is chosen at random. What is the probability that card is an ace?

2. A coach takes orders for seven of his players. Each player chooses exactly one of these items: hot dogs, nachos, chicken tenders, salad. How may different *orders* can occur? That is, do not take into account which specific player gets a given item. (When the coach arrives each player is happy!)

3. Circle R and smaller circle S are internally tangent to each other at point P and both externally tangent to line n, also at point P. A second line m cuts circle R at points B and E, circle S at points C and D, and line n at point A, so that points A, B, C, D, E occur in that order on m. If $AB = 3$, and $BC = CD = 1$, find DE.



4. Let f be a function continuous on the interval $[0, 10]$. Suppose that the average value of f on the interval $[0, 10]$ is six and the average value of f on the interval $[6, 10]$ is four. Find the average value of f on the interval $[0, 6]$.

5. Find the value of the constant a which minimizes the value of

$$\int_{-1}^{1} (x^3 - ax)^2 \, dx$$

6. For which certain values of y must solutions of the differential equation

$$\frac{dy}{dx} = y(y-1)(y-2)$$

have a point of inflection (assuming these solutions pass through a point with such a y coordinate)?

7. Let a, b, c, and d, be four distinct integers whose product is 1024. Find the least possible value of their sum.

8. Andy the ant starts at a certain vertex of a cube and randomly chooses which of the three edges that meet that vertex to travel on to the next vertex. When Andy reaches that next vertex he again randomly chooses which of the three edges to take. This process continues on and on. What is the probability that after Andy's fourth choice of edge is traversed, he has returned to the vertex from which he began his random walk?

9. Rectangle $ABCD$ is a rectangular billiard table with $AD = 1$ and $AB = 3$. A ball is at point P on side CD with $DP = 1$. A player aims the ball at point Q on side BC so that after three carooms (bounces) the ball will be headed back to where it started. Find CQ.

10. A solid is constructed on a square of side length one so that cross-sections perpendicular to one of the square's diagonals are semicircles. Find the volume of this solid.

11. Find the four-dimensional volume of the hypersphere of radius one given by the relation

$$x^2 + y^2 + z^2 + w^2 \leq 1$$

12. Evaluate

$$\int_5^{\infty} \frac{1}{\lfloor x \rfloor} - \frac{1}{\lfloor x+1 \rfloor}\, dx$$

13. Find the coefficient of x^{10} in the Maclaurin series of

$$f(x) = \frac{x}{2x^2 + 3x + 1}$$

14. Evaluate

$$\int_0^{\pi/3} 8\cos^4 x - 8\cos^2 x + 1\, dx$$

15. Six children are on a merry-go-round with six seats, all facing the same way. They decide to change how they are seated so that each child will have a different child in front of them (all still facing the same way). In how many ways can this be done?