Fall 2012 McNabb GDCTM Contest
Pre-Algebra Solutions

NO Calculators Allowed

1. Which of these quantities is the greatest?

   (A) $\frac{3}{7}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{7}{15}$ **(D)** $\frac{11}{19}$ **(E)** $\frac{13}{27}$

   Solution: D All are less than or equal to $1/2$ except for $11/19$.

2. How many millions are in a trillion?

   (A) 3 **(B)** 10^2 **(C)** 10^3 **(D)** 10^5 **(E)** 10^6

   Solution: E $10^{12}/10^6 = 10^6$

3. How many numbers are in the list

   $$21, 13, 5, -3, -11, \cdots, -203, -211$$

   where each number is 8 less than the one before it?

   (A) 20 **(B)** 23 **(C)** 28 **(D)** 29 **(E)** 30

   Solution:E Subtract 29 from each number then divide by -8. Now the list reads $1, 2, \cdots, 30$ so there are 30 numbers in the list.

4. If each of 32 boys and 32 girls receives 32 gifts then how many gifts in total were received?

   (A) 2^{10} **(B)** 2^{11} **(C)** 2^{12} **(D)** 2^{13} **(E)** 2^{15}

   Solution:B $(32 + 32)32 = 64 \cdot 32 = 2^6 \cdot 2^5 = 2^{11}$

5. A gas tank went from $3/8$ ths full to $2/3$ rds full by adding seven gallons of gas. How many more gallons must now be added to completely fill the tank?

   (A) 6 **(B)** 7 **(C)** 8 **(D)** 11 **(E)** 24

   Solution:C $2/3 - 3/8 = 7/24$ so the tank holds 24 gallons and one-third of that is 8.

6. What is the largest possible value of the greatest common factor of six different two-digit whole numbers?

   (A) 10 **(B)** 12 **(C)** 15 **(D)** 16 **(E)** 19

   Solution: D The six different two-digit numbers must all differ by multiples of their gcf, so the most efficient packing of these numbers occurs when they are consecutive multiples of their gcf, starting with the gcf. Then the two-digit multiples of 16 work: $\{16, 32, 48, 64, 80, 96\}$. So 16 is the largest as $17 \cdot 6 > 99$.

7. If $a = 5$ and $b = 3$, then the value of $4 - b(3 - a)$ is

   (A) -2 **(B)** 3 **(C)** 5 **(D)** 10 **(E)** 21

   Solution:D $4 - 3(3 - 5) = 4 - 3(-2) = 4 + 6 = 10$

8. On Black Friday a store reduced its price on a camera by 30%. Two weeks later, the item still not having sold, the store reduced the Black Friday sale price by 50%. The final price on the camera is what per cent of its original price?

   (A) 20 **(B)** 35 **(C)** 50 **(D)** 65 **(E)** 80

   Solution:B $(.7)(.5) = .35$

9. The smallest prime greater than 120 is equal to

   (A) 127 **(B)** 129 **(C)** 131 **(D)** 133 **(E)** 137

   Solution: A 121 is divisible by 11, 123 by 3, 125 by 5. But 127 is prime as it is not divisible by 2,3,5,7, 11, or 13.

10. In the sequence of numbers

$$a, b, 1, -1, 0, -1, -1, -2, \cdots$$

   each number after the second is the sum of the previous two numbers. Find the value of a.

   (A) -1 **(B)** 3 **(C)** 0 **(D)** 4 **(E)** 1

   Solution: B So $b + 1 = -1$, thus $b = -2$. Then $a + (-2) = 1$ and $a = 3$.

11. If $\frac{a}{b} = \frac{17}{4}$, $\frac{b}{c} = \frac{3}{7}$, $\frac{c}{d} = \frac{8}{17}$, and $\frac{d}{e} = \frac{7}{6}$, what is the value of $\frac{a}{e}$?

   (A) $1/34$ **(B)** $1/2$ **(C)** 1 **(D)** 2 **(E)** 14

   Solution: C The product of all the fractions is both a/e and 1.

12. In how many ways can the letters in CHEETAH be arranged so that no two consecutive letters are the same?

(A) 660 **(B)** 540 **(C)** 1260 **(D)** 564 **(E)** 330

Solution: A Principle of Inclusion/Exclusion: $7!/(2!2!) - 6!/2! - 6!/2! + 5! = 660$.

13. In Hezy's piggy bank, the value of all the pennies equals the value of all the nickels; the value of all the dimes is twice the value of all the nickels. If Hezy has only pennies, nickels, and dimes, and he has 210 coins total in his piggy bank, how much are all those coins worth?

(A) \$5.45 **(B)** \$6.00 **(C)** \$7.60 **(D)** \$8.00 **(E)** \$10.50

Solution: B Let n be the number of nickels. Then $5n$ is the number of pennies and n is also the number of dimes. So $n + 5n + n = 7n = 210$ and $n = 30$. So 30 nickels are worth \$1.50 while 150 pennies is also worth \$1.50 and 30 dimes is worth \$3.00. Thus the entire collection is worth six dollars.

14. What is the smallest positive integer n that satisfies $17n - 31m = 1$ if m must also be a positive integer?

(A) 44 **(B)** 17 **(C)** 15 **(D)** 13 **(E)** 11

Solution: E By Euclidean Algorithm or otherwise, $11 \cdot 17 - 6 \cdot 31 = 1$. Here $n = 11$ and $m = 6$. To make n smaller it would have to decrease in multiples of 31 which would force it be negative. So 11 is the smallest positive value of n where m is positive too.

15. Sixty points are equally spaced entirely around a circle. How many regular polygons can be formed using these and only these points as vertices?

(A) 60 **(B)** 68 **(C)** 78 **(D)** 88 **(E)** 89

Solution: C Add all the divisors of 60 except for the two largest as they do not correspond to actual polygons.

16. The integer 8027 is the product of exactly two primes. What is the sum of the digits of the larger of these two primes?

(A) 10 **(B)** 13 **(C)** 16 **(D)** 17 **(E)** 18

Solution: C For pre-algebra, keep dividing by primes until you see 23 goes in. For algebra one and up, use sum of cubes: $8027 = 20^3 + 3^3 = (20 + 3)(20^2 - 60 + 9) = 23 \cdot 349$. But 349 is prime, so $3 + 4 + 9 = 16$

17. Cheryl and Matthew take turns removing chips from a pile of 101 chips. On each turn they must remove 1, 2, 3, 4, or 5 chips (which of these number of chips is up to them and can change or not from turn to turn). The winner is the person who removes the last chip or chips. If Cheryl goes first, how many chips should she remove to guarantee that she will win with best play, no matter how Matthew moves?

    (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

    Solution: E Cheryl should take 5 chips on her first move to reduce the pile to 96 chips. a multiple of 6. No matter what Matthew takes on his turn, Cheryl takes the complement in terms of six, leaving Matthew with a pile of 90 chips. In this way she guarantees that Matthew will eventually face a pile of 6 chips. Whatever he takes, she can take the last remaining chips, thereby winning. If she does not take 5 chips, Matthew will turn the table on her and leave her with 96 chips! So she must take 5 on her first turn in order to win.

18. A frog is on a number line and can jump either one unit to the left or one unit to the right. If it starts at the origin and jumps randomly 6 times, what is the probability it is back at the origin at the end of those 6 jumps?

    (A) $\frac{1}{64}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{17}{32}$ **(E)** $\frac{5}{16}$

    Solution: E $\binom{6}{3}/2^6 = 5/16$

Fall 2012 McNabb GDCTM Contest
Algebra One Solutions

NO Calculators Allowed

1. On Black Friday a store reduced its price on a camera by 30%. Two weeks later, the item still not having sold, the store reduced the Black Friday sale price by 50%. The final price on the camera is what per cent of its original price?

   (A) 20 **(B)** 35 **(C)** 50 **(D)** 65 **(E)** 80

   Solution:B $(.7)(.5) = .35$

2. If one defines
   $$(a, b) \wedge (c, d) = ad - bc$$
   solve this equation for x: $(2, x) \wedge (7, -4) = 3$

   (A) $-\frac{7}{11}$ **(B)** $\frac{11}{7}$ **(C)** $\frac{7}{11}$ **(D)** 11 **(E)** $-\frac{11}{7}$

   Solution: E $-8 - 7x = 3$ leads to $x = -11/7$

3. In the sequence of numbers
   $$a, b, 1, -1, 0, -1, -1, -2, \cdots$$
   each number after the second is the sum of the previous two numbers. Find the value of a.

   (A) -1 **(B)** 3 **(C)** 0 **(D)** 4 **(E)** 1

   Solution: B So $b + 1 = -1$, thus $b = -2$. Then $a + (-2) = 1$ and $a = 3$.

4. A certain triangle in the coordinate plane has area 6. Then the x coordinates of each vertex of this triangle are doubled, but the y coordinates of each vertex are left alone. What is the area of this new triangle?

   (A) 3 **(B)** 6 **(C)** 12 **(D)** 24 **(E)** cannot be determined

   Solution: C One way: in the classic shoestring method to find the area each product will be doubled. So the overall area will be doubled.

5. If $\frac{a}{b} = \frac{17}{4}$, $\frac{b}{c} = \frac{3}{7}$, $\frac{c}{d} = \frac{8}{17}$, and $\frac{d}{e} = \frac{7}{6}$, what is the value of $\frac{a}{e}$?

(A) 1/34 **(B)** 1/2 **(C)** 1 **(D)** 2 **(E)** 14

Solution: C The product of all the fractions is both a/e and 1.

6. In how many ways can the letters in CHEETAH be arranged so that no two consecutive letters are the same?

(A) 660 **(B)** 540 **(C)** 1260 **(D)** 564 **(E)** 330

Solution: A Principle of Inclusion/Exclusion: $7!/(2!2!) - 6!/2! - 6!/2! + 5! = 660$.

7. The points x, x^2, and x^3 are graphed on the number line below. Which could be the value of x?

(A) -2 **(B)** -1 **(C)** -1/2 **(D)** 1/3 **(E)** 2

Solution: A Note that $-8 < -2 < 4$. The others do not work.

8. What is the smallest positive integer n that satisfies $17n - 31m = 1$ if m must also be a positive integer?

(A) 44 **(B)** 17 **(C)** 15 **(D)** 13 **(E)** 11

Solution: E By Euclidean Algorithm or otherwise, $11 \cdot 17 - 6 \cdot 31 = 1$. Here $n = 11$ and $m = 6$. To make n smaller it would have to decrease in multiples of 31 which would force it be negative. So 11 is the smallest positive value of n where m is positive too.

9. In how many ways can 9 students be divided into 3 groups of 3 students each?

(A) 81 **(B)** 180 **(C)** 280 **(D)** 540 **(E)** 1680

Solution: C $(\binom{9}{3} \cdot \binom{6}{3} \cdot \binom{3}{3})/3! = 280$ because the groups are unordered.

10. How many solutions does the equation $|x - 2| = |4 - x|$ have?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** infinitely many

Solution:B Only $x = 3$ is equidistant from 2 and 4.

11. Which of the integers below can be expressed in the form $p^2+q^2+r^2+s^2+t^2$ where p, q, r, s and t are all odd integers?

(A) 2012 **(B)** 2013 **(C)** 2014 **(D)** 2015 **(E)** 2016

Solution: B The square of an odd is one greater than a multiple of 4. So the sum of 5 such is also one greater than a multiple of 4. This eliminates each answer except 2013 and 2013 can be written in such a manner.

12. Sixty points are equally spaced entirely around a circle. How many regular polygons can be formed using these and only these points as vertices?

(A) 60 **(B)** 68 **(C)** 78 **(D)** 88 **(E)** 89

Solution: C Add all the divisors of 60 except for the two largest as they do not correspond to actual polygons.

13. Cheryl and Matthew take turns removing chips from a pile of 101 chips. On each turn they must remove 1, 2, 3, 4, or 5 chips (which of these number of chips is up to them and can change or not from turn to turn). The winner is the person who removes the last chip or chips. If Cheryl goes first, how many chips should she remove to guarantee that she will win with best play, no matter how Matthew moves?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Solution: E Cheryl should take 5 chips on her first move to reduce the pile to 96 chips. a multiple of 6. No matter what Matthew takes on his turn, Cheryl takes the complement in terms of six, leaving Matthew with a pile of 90 chips. In this way she guarantees that Matthew will eventually face a pile of 6 chips. Whatever he takes, she can take the last remaining chips, thereby winning. If she does not take 5 chips, Matthew will turn the table on her and leave her with 96 chips! So she must take 5 on her first turn in order to win.

14. A problem from the *Liber Abaci*, a math text written by Fibonnaci in the 13th century:

> On a certain ground there are two towers, one of which is 30 feet high, the other 40, and they are only 50 feet apart; two birds descending together from the heights of the two towers fly to the center of a fountain between the towers; the distance from the center [of the fountain] to the foot of the higher tower is sought.

In this problem assume: the birds are flying at the same speed, depart at the same time, and arrive together at the fountain; and the fountain and feet of the towers are collinear.

(A) 18 **(B)** 20 **(C)** 22 **(D)** 24 **(E)** 32

Solution: A Let $x =$ the distance from the center of the fountain to the base of the higher tower. Then $50 - x =$ the distance from the center of the fountain to the base of the lower tower. By the Pythagorean Theorem $x^2 + 40^2 = (50 - x)^2 + 30^2$ or $x^2 + 1600 = x^2 - 100x + 2500 + 900$ or $100x = 1800$ or $x = 18$.

15. A boat goes downriver from A to B in 3 days and returns upriver from B to A in 4 days. How long in days would it take an inner tube to float downriver from A to B?

(A) 12 **(B)** 18 **(C)** 24 **(D)** 30 **(E)** 32

Solution: C Let $r =$rate of boat in still water and $c =$rate of the current. Take the distance from A to B as unit distance. Then $1/(r + c) = 3$ and $1/(r - c) = 4$. Then $r + c = 1/3$ and $r - c = 1/4$. Subtracting these equations yields $2c = 1/12$ or $c = 1/24$. Thus it would take 24 days for the tube to float from A to B.

16. Find the value of x if

$$\begin{aligned} 3x + 2y - z &= 1 \\ -x + y - 3z &= 7 \\ x + 2y + 9z &= -1 \end{aligned}$$

(A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

Solution:A Take -3 times the first equation and add it to 4 times the second, then add that result to the last equation to get $-12x = 24$ or $x = -2$.

17. A frog is on a number line and can jump either one unit to the left or one unit to the right. If it starts at the origin and jumps randomly 6 times, what is the probability it is back at the origin at the end of those 6 jumps?

(A) $\frac{1}{64}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{17}{32}$ **(E)** $\frac{5}{16}$

Solution: E $\binom{6}{3}/2^6 = 5/16$

18. The coefficient of x^{18} in the product

$$(x+1)(x+3)(x+5)(x+7)\cdots(x+37)$$

is equal to

(A) 1 **(B)** 243 **(C)** 361 **(D)** 400 **(E)** 401

Solution: C As there are 19 odds from 1 to 37, we want the sum of these odds, which is $19^2 = 361$.

Fall 2012 McNabb GDCTM Contest
Geometry Solutions

NO Calculators Allowed

1. If one defines
$$(a, b) \wedge (c, d) = ad - bc$$
solve this equation for x: $(2, x) \wedge (7, -4) = 3$

   (A) $-\dfrac{7}{11}$ **(B)** $\dfrac{11}{7}$ **(C)** $\dfrac{7}{11}$ **(D)** 11 **(E)** $-\dfrac{11}{7}$

   Solution: E $-8 - 7x = 3$ leads to $x = -11/7$

2. A certain triangle in the coordinate plane has area 6. Then the x coordinates of each vertex of this triangle are doubled, but the y coordinates of each vertex are left alone. What is the area of this new triangle?

   (A) 3 **(B)** 6 **(C)** 12 **(D)** 24 **(E)** cannot be determined

   Solution: C One way: in the classic shoestring method to find the area each product will be doubled. So the overall area will be doubled.

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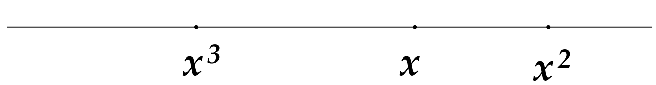


   Solution: A Note that $-8 < -2 < 4$. The others do not work.

4. In how many ways can the letters in CHEETAH be arranged so that no two consecutive letters are the same?

   (A) 660 **(B)** 540 **(C)** 1260 **(D)** 564 **(E)** 330

   Solution: A Principle of Inclusion/Exclusion: $7!/(2!2!) - 6!/2! - 6!/2! + 5! = 660$.

5. What is the area of a rhombus with sides equal to 13 and short diagonal equal to 10?

   (A) 60 **(B)** 65 **(C)** 120 **(D)** 130 **(E)** 260

   Solution: C Use the $(5, 12, 13)$ Pythagorean Triple to deduce that the long diagonal has length 24. So the area is half the product of the diagonals, or 120.

6. In how many ways can 9 students be divided into 3 groups of 3 students each?

   (A) 81 **(B)** 180 **(C)** 280 **(D)** 540 **(E)** 1680

   Solution: C $(\binom{9}{3} \cdot \binom{6}{3} \cdot \binom{3}{3})/3! = 280$ because the groups are unordered.

7. In $\triangle ABC$, points D and E lie on sides AC and AB respectively. Draw BD and CE intersecting at point F. Suppose $AC = AB = 12$, $BF = FC = 6$, and $EF = FD = 5$. Find the length of AD.

   (A) 7 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

   Solution:C By similar triangles $DE = (5/6)BC$. By another pair of similar triangles, $AD = (5/6)12 = 10$. Alternatively, use Menelaus' Theorem. A third way is mass points.

8. On the first test of the school year an algebra class averaged 81. If the three lowest scoring exams were not considered, the average would have been 84. If those three lowest scores were 52, 62, and 66, how many students are in the algebra class?

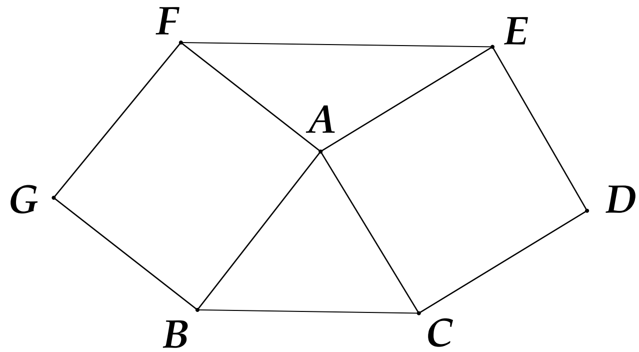
   (A) 21 **(B)** 24 **(C)** 26 **(D)** 27 **(E)** 28

   Solution: B Let n be the number of students in the class. Then using total points, $81n = 84(n-3) + 52 + 62 + 66$ or $81n = 84n - 72$ or $3n = 72$ or $n = 24$.

9. An equilateral triangle ABC fits between two squares, $AEDC$ and $ABGF$ as shown. Segment FE is drawn. What is the measure of $\angle AFE$ in degrees?

   (A) 30 **(B)** 45 **(C)** 50 **(D)** 55 **(E)** 60

   Solution: A Since the angles around A must sum to 360 degrees, it follows that $\angle FAE = 120$. But since $\triangle FAE$ is isosceles, we have right away that $\angle AFE = 30$.



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11. Sixty points are equally spaced entirely around a circle. How many regular polygons can be formed using these and only these points as vertices?

    (A) 60 **(B)** 68 **(C)** 78 **(D)** 88 **(E)** 89

    Solution: C Add all the divisors of 60 except for the two largest as they do not correspond to actual polygons.

12. There are two non-congruent triangles ABC with $AB = 8$, $BC = 5$, and $\angle A = 30^\circ$. What is the positive difference of their areas?

    (A) 5 **(B)** 6 **(C)** $3\sqrt{3}$ **(D)** $5\sqrt{3}$ **(E)** 12

    Solution: E The difference in the areas is the area of an isosceles triangle with equal sides 5 and altitude 4. So it has base 6 and area 12.

13. The coefficient of x^{18} in the product

$$(x+1)(x+3)(x+5)(x+7)\cdots(x+37)$$

is equal to

(A) 1 **(B)** 243 **(C)** 361 **(D)** 400 **(E)** 401

Solution: C As there are 19 odds from 1 to 37, we want the sum of these odds, which is $19^2 = 361$.

14. Let $a \neq b$. The equation of the perpendicular bisector of the segment with endpoints (a, b) and (b, a) is

(A) $y = x$ **(B)** $y = 0$ **(C)** $x = 0$ **(D)** $y = -x$ **(E)** $y = 2x$

Solution: A Point (b, a) is the reflection of (a, b) across $y = x$.

15. Suppose that the statements:

No *zoofs* are *zarns*
At least one *zune* is not a *zoof*

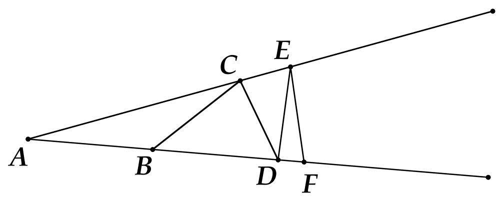
are true. Which of the following must be true?

(A) At least one *zune* is a *zoof*
(B) No *zarn* is a *zune*
(C) At least one *zarn* is not a *zune*
(D) All *zunes* are *zarns*
(E) None of the above

Solution:E All three categories could be pairwise disjoint. And *zunes* and *zarns* can have any possible relationship. So none of (A) through (D) must be true. Thus (E) holds.

16. Points B, C, D, E, and F lie as shown on $\angle A$ with $AB = BC = CD = DE = EF$ as shown. If $\angle AEF$ is right, then find, in degrees, the measure of $\angle CAB$.

(A) 16 **(B)** 18 **(C)** 20 **(D)** 22 **(E)** 24



Solution:B Let $\angle CAB = x$. Then by the Isosceles Triangle Theorem and the Exterior Angle Theorem used as needed, $3x + 180 - 8x = 90$ or $5x = 90$ or $x = 18 = \angle CAB$.

17. The line $4x - 8y = 15$ is irritating to graph as it contains no points of the form (a, b), with both a and b integers. Such points are called *lattice points*. What is the minimum distance between this line and the set of lattice points?

(A) $\frac{1}{12}$ **(B)** $\frac{1}{10}$ **(C)** $\frac{\sqrt{5}}{20}$
(D) $\frac{\sqrt{5}}{15}$ **(E)** no minimum exists

Solution: C The distance from a point (a, b) to this line is given by $|4a - 8b - 15|/\sqrt{80}$. The smallest that $|4a - 8b - 15|$ could when a and b are integers is 1 (because $4a - 8b$ is a multiple of 4 and 15 is not), and this occurs when $a = 0$ and $b = 2$ for instance. So the least distance is $1/\sqrt{80} = \sqrt{5}/20$.

18. The real number $\sqrt{41 - 24\sqrt{2}}$ can be put in the form $a\sqrt{2} - b$ where a and b are positive integers. What is the value of $a + b$?

(A) 4 **(B)** 7 **(C)** 8 **(D)** 16 **(E)** 18

Solution: B Squaring the given form $(2a^2 + b^2) - 2ab\sqrt{2} = 41 - 24\sqrt{2}$, so that $2a^2 + b^2 = 41$ and $ab = 12$. Using factors of 12, do trials until seeing that $a = 4$ and $b = 3$ works. Then check that $4\sqrt{2} - 3 > 0$. So $a + b = 7$.

19. An ordered triple of positive integers (a, b, c) with $a < b < c$ and $a^2 + b^2 = c^2$ is called *Pythagorean*. Find the perimeter of the only Pythagorean triple

with $a = 11$.

(A) 88 **(B)** 99 **(C)** 121 **(D)** 132 **(E)** 154

Solution: D Then $c^2 - b^2 = 121 = (c-b)(c+b)$ If $c+b = 121$ and $c - b = 1$ then $c = 61$ and $b = 60$. This does work out with the triangle inequality. The only other possibility is $c + b = 11$ and $c - b = 11$ but this leads to $b = 0$ which is impossible. The perimeter is then $11 + 60 + 61 = 132$.

20. The trapezoid $ABCD$ has $AB \parallel CD$, $AB = 5$, and $DC = 12$. Draw EF parallel to AB with E on AD and F on BC. If EF splits trapezoid $ABCD$ into two trapezoids of equal area, what is the length of EF?

(A) 9 **(B)** $\dfrac{120}{17}$ **(C)** $\dfrac{17}{2}$ **(D)** $\dfrac{13\sqrt{2}}{2}$ **(E)** $2\sqrt{15}$

Solution: D Let $x = EF$. Let h and k be the altitudes of trapezoids $EFCD$ and $ABFE$ respectively. The equality of the areas is equivalent to $(x + 12)h = (x+5)k$ while $h/k = (12-x)/(x-5)$. From $(12-x)/(x-5) = (x+5)/(x+12)$ we obtain $x = 13\sqrt{2}/2$, which happens to (always) be the root-mean-square average of the bases.

Fall 2012 McNabb GDCTM Contest
Algebra Two Solutions

NO Calculators Allowed

1. How many numbers are in the list

$$21, 13, 5, -3, -11, \cdots, -203, -211$$

where each number is 8 less than the one before it?

(A) 20 **(B)** 23 **(C)** 28 **(D)** 29 **(E)** 30

Solution:E Subtract 29 from each number then divide by -8. Now the list reads $1, 2, \cdots, 30$ so there are 30 numbers in the list.

2. What is the largest possible value of the greatest common factor of six different two-digit whole numbers?

(A) 10 **(B)** 12 **(C)** 15 **(D)** 16 **(E)** 19

Solution: D The six different two-digit numbers must all differ by multiples of their gcf, so the most efficient packing of these numbers occurs when they are consecutive multiples of their gcf, starting with the gcf. Then the two-digit multiples of 16 work:$\{16, 32, 48, 64, 80, 96\}$. So 16 is the largest as $17 \cdot 6 > 99$.

3. In the sequence of numbers

$$a, b, 1, -1, 0, -1, -1, -2, \cdots$$

each number after the second is the sum of the previous two numbers. Find the value of a.

(A) -1 **(B)** 3 **(C)** 0 **(D)** 4 **(E)** 1

Solution: B So $b + 1 = -1$, thus $b = -2$. Then $a + (-2) = 1$ and $a = 3$.

4. A certain triangle in the coordinate plane has area 6. Then the x coordinates of each vertex of this triangle are doubled, but the y coordinates of each vertex are left alone. What is the area of this new triangle?

(A) 3 **(B)** 6 **(C)** 12 **(D)** 24 **(E)** cannot be determined

Solution: C One way: in the classic shoestring method to find the area each product will be doubled. So the overall area will be doubled.

5. The points x, x^2, and x^3 are graphed on the number line below. Which could be the value of x?

(A) -2 **(B)** -1 **(C)** -1/2 **(D)** 1/3 **(E)** 2

Solution: A Note that $-8 < -2 < 4$. The others do not work.

6. There are two non-congruent triangles ABC with $AB = 8$, $BC = 5$, and $\angle A = 30°$. What is the positive difference of their areas?

(A) 5 **(B)** 6 **(C)** $3\sqrt{3}$ **(D)** $5\sqrt{3}$ **(E)** 12

Solution: E The difference in the areas is the area of an isosceles triangle with equal sides 5 and altitude 4. So it has base 6 and area 12.

7. In how many ways can the letters in CHEETAH be arranged so that no two consecutive letters are the same?

(A) 660 **(B)** 540 **(C)** 1260 **(D)** 564 **(E)** 330

Solution: A Principle of Inclusion/Exclusion: $7!/(2!2!) - 6!/2! - 6!/2! + 5! = 660$.

8. The coefficient of x^{18} in the product

$$(x+1)(x+3)(x+5)(x+7)\cdots(x+37)$$

is equal to

(A) 1 **(B)** 243 **(C)** 361 **(D)** 400 **(E)** 401

Solution: C As there are 19 odds from 1 to 37, we want the sum of these odds, which is $19^2 = 361$.

9. On the first test of the school year an algebra class averaged 81. If the three lowest scoring exams were not considered, the average would have been 84. If those three lowest scores were 52, 62, and 66, how many students are in the algebra class?

(A) 21 **(B)** 24 **(C)** 26 **(D)** 27 **(E)** 28

Solution: B Let n be the number of students in the class. Then using total points, $81n = 84(n-3) + 52 + 62 + 66$ or $81n = 84n - 72$ or $3n = 72$ or $n = 24$.

10. In $\triangle ABC$, points D and E lie on sides AC and AB respectively. Draw BD and CE intersecting at point F. Suppose $AC = AB = 12$, $BF = FC = 6$, and $EF = FD = 5$. Find the length of AD.

(A) 7 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

Solution:C By similar triangles $DE = (5/6)BC$. By another pair of similar triangles, $AD = (5/6)12 = 10$. Alternatively, use Menelaus' Theorem. A third way is mass points.

11. If the point $(1, -5)$ lies on the graph of $y = -f(1-2x) + 2$ which point below must lie on the graph of $y = 3f(5x-6) - 8$?

(A) $(1, 13)$ **(B)** $(3, -8)$ **(C)** $(-1, -8)$ **(D)** $(0, 11)$ **(E)** $(-4, 8)$

Solution: A So $-5 = -f(-1) + 2$ or $f(-1) = 7$. Thus if $x = 1$, then $3f(5x-6) - 8 = 3f(-1) - 8 = 3 \cdot 7 - 8 = 21 - 8 = 13$.

12. How many real solutions are there to the equation

$$(x+1)(x+2)(x+3)(x+4) = (x+5)(x+6)(x+7)(x+8)$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Solution: B There is only one real solution. The two sides balance out when $x = -4.5$. The equation is a cubic so there are at most 3 real roots. Using that -4.5 is a root the cubic reduces to the quadratic $2x^2 + 9x + 23 = 0$, which has a negative discriminant. So there is only one real root.

13. How many ordered pairs (x, y) of positive integers satisfy both

$$\frac{x}{8} + \frac{y}{3} > 1 \qquad \text{and} \qquad \frac{x}{12} + \frac{y}{7} < 1$$

(A) 22 **(B)** 23 **(C)** 24 **(D)** 25 **(E)** 26

Solution: E By Pick's Theorem, the area of the quadrilateral with vertices at $(8,0), (12,0), (0,7), (0,3) = B/2 + I - 1$ where B is the number of lattice points on the boundary and I is both the number of lattice points in the interior and the number of integer solutions to our system. Thus $(1/2)(12 \cdot 7 - 3 \cdot 8) = 1/2(5+5) + I - 1$ or $30 = 4 + I$ or $I = 26$.

14. A boat goes downriver from A to B in 3 days and returns upriver from B to A in 4 days. How long in days would it take an inner tube to float downriver from A to B?

(A) 12 **(B)** 18 **(C)** 24 **(D)** 30 **(E)** 32

Solution: C Let r =rate of boat in still water and c =rate of the current. Take the distance from A to B as unit distance. Then $1/(r+c) = 3$ and $1/(r-c) = 4$. Then $r+c = 1/3$ and $r-c = 1/4$. Subtracting these equations yields $2c = 1/12$ or $c = 1/24$. Thus it would take 24 days for the tube to float from A to B.

15. Which of the following are true for all positive real numbers a and b?

I. $\sqrt{ab} = \sqrt{a}\sqrt{b}$

II. $\sqrt{a} + \sqrt{b} > \sqrt{a+b}$

III. $\sqrt{\dfrac{a^2+b^2}{2}} < \dfrac{a+b}{2}$

(A) I only **(B)** I and II only **(C)** I and III only
(D) II and III only **(E)** I, II, and III

Solution: B Roman I is true. Roman II is true by squaring both sides. Roman III is false when $a = b$.

16. Three congruent circles are mutually externally tangent to each other and each circle is tangent to a different side of an equilateral triangle of side length 2. The common radius of the circles can be written in the form $a - \sqrt{b}$, where a and b are positive integers. What is the value of $3a + b$?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

Solution: C Let A be a vertex of the equilateral triangle, B its centroid, and C the midpoint of a side adjacent to A. Then $\triangle ABC$ is a 30-60-90 degree triangle, with $\angle BAC = 30^\circ$, $AC = 1$, and $AB = 2/\sqrt{3}$. Let D be the center of the circle tangent to the triangle at C. Then AB is tangent to this circle at some point, call it E (a point where circle D is tangent also to one of the other circles). By Equality of Tangents, $AE = 1$. Note that $\triangle DEB$ is right, with $\angle EBD = 60^\circ$ and $BE = (2/\sqrt{3}) - 1$. Thus $\tan 60^\circ = ED/((2/\sqrt{3}) - 1) = \sqrt{3}$ which implies $r = ED = ((2/\sqrt{3}) - 1)\sqrt{3} = 2 - \sqrt{3}$. Thus $a = 2$, $b = 3$, and $3a + b = 9$.

17. Find a number c so that the three distinct solutions $x_1 < x_2 < x_3$ of the equation $x^3 + 6x^2 - 8x + 4 = c$ satisfy $x_1 + x_3 = 2x_2$.

(A) 36 **(B)** 37 **(C)** 38 **(D)** 39 **(E)** 40

Solution: A Since $x_1 + x_2 + x_3 = -6 = 3x_2$, then $x_2 = -2$ is a root. Then $(-2)^3 + 6(-2)^2 - 8(-2) + 4 = -8 + 24 + 16 + 4 = 36 = c$. And in this case indeed all three roots are real and $x_2 = -2$ is in the middle.

18. Find the value of x if

$$\begin{aligned} 3x + 2y - z &= 1 \\ -x + y - 3z &= 7 \\ x + 2y + 9z &= -1 \end{aligned}$$

(A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

Solution:A Take -3 times the first equation and add it to 4 times the second, then add that result to the last equation to get $-12x = 24$ or $x = -2$.

19. At how many points do the graphs of $y = x^4 + x^3 - 2x^2 + x$ and $y = x$ intersect?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Solution: C Set these functions equal and solve for x. There are three solutions, $x = 0, -2, 1$. Thus the graphs intersect three times.

20. A caravan of 5 cars is headed down I-35 for the annual Texas A&M High School Math Tournament. In how many ways can the caravan be re-formed after a rest stop so that each non-leading car has a different car in front of it from the one it had before the rest stop? Note the car that lead before the rest stop may still be leading after it and also could be any of the following cars after it as well.

(A) 31 **(B)** 53 **(C)** 60 **(D)** 65 **(E)** 71

Solution: B Use the Principle of Inclusion/Exclusion to obtain the answer as $5! - 4 \cdot 4! + 3 \cdot 4!/2! - 2 \cdot 4!/3! + 4!/4! = 53$. If $ABCDE$ is the order going down with A in front then there are four violations AB, BC, CD, and DE so we alternately remove and add the number of times they occur at least once, at least twice, etc...

Fall 2012 McNabb GDCTM Contest
PreCalculus Solutions

NO Calculators Allowed

1. If one defines

$$(a, b) \wedge (c, d) = ad - bc$$

solve this equation for x: $(2, x) \wedge (7, -4) = 3$

(A) $-\frac{7}{11}$ **(B)** $\frac{11}{7}$ **(C)** $\frac{7}{11}$ **(D)** 11 **(E)** $-\frac{11}{7}$

Solution: E $-8 - 7x = 3$ leads to $x = -11/7$

2. Suppose that the statements:

No *zoofs* are *zarns*
At least one *zune* is not a *zoof*

are true. Which of the following must be true?

(A) At least one *zune* is a *zoof*
(B) No *zarn* is a *zune*
(C) At least one *zarn* is not a *zune*
(D) All *zunes* are *zarns*
(E) None of the above

Solution:E All three categories could be pairwise disjoint. And *zunes* and *zarns* can have any possible relationship. So none of (A) through (D) must be true. Thus (E) holds.

3. In how many ways can the letters in CHEETAH be arranged so that no two consecutive letters are the same?

(A) 660 **(B)** 540 **(C)** 1260 **(D)** 564 **(E)** 330

Solution: A Principle of Inclusion/Exclusion: $7!/(2!2!) - 6!/2! - 6!/2! + 5! = 660$.

4. The coefficient of x^{18} in the product

$$(x + 1)(x + 3)(x + 5)(x + 7) \cdots (x + 37)$$

is equal to

(A) 1 **(B)** 243 **(C)** 361 **(D)** 400 **(E)** 401

Solution: C As there are 19 odds from 1 to 37, we want the sum of these odds, which is $19^2 = 361$.

5. In how many ways can 9 students be divided into 3 groups of 3 students each?

   (A) 81 **(B)** 180 **(C)** 280 **(D)** 540 **(E)** 1680

   Solution: C $(\binom{9}{3} \cdot \binom{6}{3} \cdot \binom{3}{3})/3! = 280$ because the groups are unordered.

6. There are two non-congruent triangles ABC with $AB = 8$, $BC = 5$, and $\angle A = 30^\circ$. What is the positive difference of their areas?

   (A) 5 **(B)** 6 **(C)** $3\sqrt{3}$ **(D)** $5\sqrt{3}$ **(E)** 12

   Solution: E The difference in the areas is the area of an isosceles triangle with equal sides 5 and altitude 4. So it has base 6 and area 12.

7. If the point $(1, -5)$ lies on the graph of $y = -f(1 - 2x) + 2$ which point below must lie on the graph of $y = 3f(5x - 6) - 8$?

   (A) $(1, 13)$ **(B)** $(3, -8)$ **(C)** $(-1, -8)$ **(D)** $(0, 11)$ **(E)** $(-4, 8)$

   Solution: A So $-5 = -f(-1) + 2$ or $f(-1) = 7$. Thus if $x = 1$, then $3f(5x - 6) - 8 = 3f(-1) - 8 = 3 \cdot 7 - 8 = 21 - 8 = 13$.

8. A function of the form $f(x) = a \cdot b^x$ where a and b are constants satisfies: $f(1) = 26 - 60r$, $f(3) = 162r$, and $f(4) = 486r$. What is the value of the constant r?

   (A) $1/9$ **(B)** $1/6$ **(C)** $1/3$ **(D)** 3 **(E)** 6

   Solution: C Since $f(4)$ is 3 times $f(3)$ and 4 is one more than 3 it follows that $b = 3$. Then $a = (26 - 60r)/3 = 162r/27 = 6r$ or $26 - 60r = 18r$ or $78r = 26$ or $r = 1/3$.

9. At how many points do the graphs of $y = x^4 + x^3 - 2x^2 + x$ and $y = x$ intersect?

   (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

   Solution: C Set these functions equal and solve for x. There are three solutions, $x = 0, -2, 1$. Thus the graphs intersect three times.

10. For which values of the constant parameter a does the system below have exactly one solution?

$$\begin{aligned} x^2 + xy + y^2 &= 3 \\ x + 2y &= a \end{aligned}$$

(A) 0 **(B)** $\pm\sqrt{6}$ **(C)** 3 **(D)** $\pm\sqrt{12}$ **(E)** 2

Solution: D Substitute $x = a - 2y$ into the top equation to get $a^2 - 4ay + 4y^2 + ay - 2ay = 2y^2 + y^2 = 3$ or $3y^2 - 3ay + (a^2 - 3) = 0$. Set the discriminant $D = 0$, that is $9a^2 - 4 \cdot 3 \cdot (a^2 - 3) = 0$ or $36 - 3a^2 = 0$ or $a^2 = 12$.

11. Chords AB and CD of circle R intersect at a right angle at point P. If $AP = 3$, $PB = 8$, and $CP = 6$, what is the diameter of circle R?

(A) $8\sqrt{2}$ **(B)** 10 **(C)** $6\sqrt{3}$ **(D)** 11 **(E)** $5\sqrt{5}$

Solution: Use Power of a Point to deduce that $PD = 4$. By a neat theorem, $3^2 + 8^2 + 6^2 + 4^2 = d^2$ or $125 = d^2$ so $d = 5\sqrt{5}$. Or else just use coordinates setting the origin to be at point P.

12. The graph of $3x^2 + 4xy + 3y^2 = 50$ is an ellipse. What is the greatest distance of any of its points to the origin?

(A) $\sqrt{\frac{50}{3}}$ **(B)** $\sqrt{50}$ **(C)** 5 **(D)** 6 **(E)** $\sqrt{\dfrac{25}{3}}$

Solution: B Since the center of the ellipse is the origin and $x = y$ is an axis of symmetry of the equation it and its perpendicular $x = -y$ must contain in some order the major and minor axes. The line $y = x$ intersects the ellipse at the points $\pm(\sqrt{5}, \sqrt{5})$ while the line $y = -x$ intersects the ellipse at the points $\pm(5, -5)$, which are further from the origin. Thus the major semi-axis equals $\sqrt{50}$, the answer to the problem.

13. A caravan of 5 cars is headed down I-35 for the annual Texas A&M High School Math Tournament. In how many ways can the caravan be re-formed after a rest stop so that each non-leading car has a different car in front of it from the one it had before the rest stop? Note the car that lead before the rest stop may still be leading after it and also could be any of the following cars after it as well.

(A) 31 **(B)** 53 **(C)** 60 **(D)** 65 **(E)** 71

Solution: B Use the Principle of Inclusion/Exclusion to obtain the answer as $5! - 4 \cdot 4! + 3 \cdot 4!/2! - 2 \cdot 4!/3! + 4!/4! = 53$. If $ABCDE$ is the order going down with A in front then there are four violations AB, BC, CD, and DE so we alternately remove and add the number of times they occur at least once, at least twice, etc...

14. If $a^2 + b^2 = 25$, what is the minimum possible value of $a + b$?

(A) -10 **(B)** $-5\sqrt{2}$ **(C)** 0 **(D)** $5\sqrt{2}$ **(E)** 10

Solution: B The value of $a + b$ will be at a minimum for the least c for which $a + b = c$ still intersects the circle. This will occur when $a + b = c$ is a tangent to the circle in the third quadrant, so at the point $(-5/\sqrt{2}, -5/\sqrt{2}$. At that point then $a + b = -5\sqrt{2}$.

15. In isosceles triangle ABC with $AB = AC$ the angle bisectors of $\angle B$ and $\angle C$ meet the altitude AD at Q. If $\angle BAC = 2\theta$, what is the ratio $QD : AD$?

(A) $\dfrac{\sin\theta}{1+\sin\theta}$ **(B)** $\dfrac{1}{3}$ **(C)** $\dfrac{\sin\theta}{2}$ **(D)** $1 - \sin\theta$ **(E)** $\tan\theta$

Solution: A Let $\alpha = 90 - \theta$. Note that $DQ/DC = \tan(\frac{\alpha}{2}) = \sin\alpha/(1 + \cos\alpha)$ and $AD/DC = \tan\alpha$, so that $DQ/AB = \cos\alpha/(1 + \cos\alpha) = \sin\theta/(1 + \sin\theta)$.

16. How many ordered pairs (x, y) of positive integers satisfy both

$$\frac{x}{8} + \frac{y}{3} > 1 \qquad \text{and} \qquad \frac{x}{12} + \frac{y}{7} < 1$$

(A) 22 **(B)** 23 **(C)** 24 **(D)** 25 **(E)** 26

Solution: E By Pick's Theorem, the area of the quadrilateral with vertices at $(8,0), (12,0), (0,7), (0,3) = B/2 + I - 1$ where B is the number of lattice points on the boundary and I is both the number of lattice points in the interior and the number of integer solutions to our system. Thus $(1/2)(12 \cdot 7 - 3 \cdot 8) = 1/2(5+5) + I - 1$ or $30 = 4 + I$ or $I = 26$.

17. Let S_n be the sum of the first n terms of a certain arithmetic sequence. If $S_{12} = 8$ and $S_{24} = 20$, then find the value of S_{36}.

    (A) 28 **(B)** 36 **(C)** 40 **(D)** 42 **(E)** 48

    Solution: B We have $a_1 + a_{12} = 4/3$ and $a_1 + a_{24} = 5/3$. Subtracting $a_{24} - a_{12} = 1/3 = 12d$ so $d = 1/36$. And $a_1 + a_{36} = a_1 + a_{24} + 12d = 5/3 + 1/3 = 2$ so $S_{36} = 1 \cdot 36 = 36$.

18. A parallelogram has sides of length 7 and 9. Its longer diagonal has length 14. What is the length of its shorter diagonal?

    (A) 8 **(B)** 8.5 **(C)** 9 **(D)** 9.5 **(E)** 10

    Solution: A By the Parallelogram Law, $14^2 + d^2 = 2(7^2 + 9^2) = 260$ or $d^2 = 64$ or $d = 8$, where d is the other diagonal, in this case shorter. Or one can use the Law of Cosines twice.

19. Find a number c so that the three distinct solutions $x_1 < x_2 < x_3$ of the equation $x^3 + 6x^2 - 8x + 4 = c$ satisfy $x_1 + x_3 = 2x_2$.

    (A) 36 **(B)** 37 **(C)** 38 **(D)** 39 **(E)** 40

    Solution: A Since $x_1 + x_2 + x_3 = -6 = 3x_2$, then $x_2 = -2$ is a root. Then $(-2)^3 + 6(-2)^2 - 8(-2) + 4 = -8 + 24 + 16 + 4 = 36 = c$. And in this case indeed all three roots are real and $x_2 = -2$ is in the middle.

20. How many real solutions are there to the equation

$$(x+1)(x+2)(x+3)(x+4) = (x+5)(x+6)(x+7)(x+8)$$

    (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

    Solution: B There is only one real solution. The two sides balance out when $x = -4.5$. The equation is a cubic so there are at most 3 real roots. Using that -4.5 is a root the cubic reduces to the quadratic $2x^2 + 9x + 23 = 0$, which has a negative discriminant. So there is only one real root.

Fall 2012 McNabb GDCTM Contest
Calculus Solutions

NO Calculators Allowed

Assume all variables are real unless otherwise stated in the problem.

1. If $\frac{a}{b} = \frac{17}{4}$, $\frac{b}{c} = \frac{3}{7}$, $\frac{c}{d} = \frac{8}{17}$, and $\frac{d}{e} = \frac{7}{6}$, what is the value of $\frac{a}{e}$?

   (A) $1/34$ **(B)** $1/2$ **(C)** 1 **(D)** 2 **(E)** 14

   Solution: C The product of all the fractions is both a/e and 1.

2. In how many ways can the letters in CHEETAH be arranged so that no two consecutive letters are the same?

   (A) 660 **(B)** 540 **(C)** 1260 **(D)** 564 **(E)** 330

   Solution: A Principle of Inclusion/Exclusion: $7!/(2!2!) - 6!/2! - 6!/2! + 5! = 660$.

3. The coefficient of x^{18} in the product

$$(x+1)(x+3)(x+5)(x+7)\cdots(x+37)$$

   is equal to

   (A) 1 **(B)** 243 **(C)** 361 **(D)** 400 **(E)** 401

   Solution: C As there are 19 odds from 1 to 37, we want the sum of these odds, which is $19^2 = 361$.

4. What is the smallest positive integer n that satisfies $17n - 31m = 1$ if m must also be a positive integer?

   (A) 44 **(B)** 17 **(C)** 15 **(D)** 13 **(E)** 11

   Solution: E By Euclidean Algorithm or otherwise, $11 \cdot 17 - 6 \cdot 31 = 1$. Here $n = 11$ and $m = 6$. To make n smaller it would have to decrease in multiples of 31 which would force it be negative. So 11 is the smallest positive value of n where m is positive too.

5. A boat goes downriver from A to B in 3 days and returns upriver from B to A in 4 days. How long in days would it take an inner tube to float downriver from A to B?

(A) 12 **(B)** 18 **(C)** 24 **(D)** 30 **(E)** 32

Solution: C Let r =rate of boat in still water and c =rate of the current. Take the distance from A to B as unit distance. Then $1/(r+c) = 3$ and $1/(r-c) = 4$. Then $r+c = 1/3$ and $r-c = 1/4$. Subtracting these equations yields $2c = 1/12$ or $c = 1/24$. Thus it would take 24 days for the tube to float from A to B.

6. On the first test of the school year an algebra class averaged 81. If the three lowest scoring exams were not considered, the average would have been 84. If those three lowest scores were 52, 62, and 66, how many students are in the algebra class?

(A) 21 **(B)** 24 **(C)** 26 **(D)** 27 **(E)** 28

Solution: B Let n be the number of students in the class. Then using total points, $81n = 84(n-3) + 52 + 62 + 66$ or $81n = 84n - 72$ or $3n = 72$ or $n = 24$.

7. Suppose that the statements:

No *zoofs* are *zarns*
At least one *zune* is not a *zoof*

are true. Which of the following must be true?

(A) At least one *zune* is a *zoof*
(B) No *zarn* is a *zune*
(C) At least one *zarn* is not a *zune*
(D) All *zunes* are *zarns*
(E) None of the above

Solution:E All three categories could be pairwise disjoint. And *zunes* and *zarns* can have any possible relationship. So none of (A) through (D) must be true. Thus (E) holds.

8. Cheryl and Matthew take turns removing chips from a pile of 101 chips. On each turn they must remove 1, 2, 3, 4, or 5 chips (which of these number of chips is up to them and can change or not from turn to turn). The winner is the person who removes the last chip or chips. If Cheryl goes first, how many chips should she remove to guarantee that she will win with best

play, no matter how Matthew moves?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Solution: E Cheryl should take 5 chips on her first move to reduce the pile to 96 chips. a multiple of 6. No matter what Matthew takes on his turn, Cheryl takes the complement in terms of six, leaving Matthew with a pile of 90 chips. In this way she guarantees that Matthew will eventually face a pile of 6 chips. Whatever he takes, she can take the last remaining chips, thereby winning. If she does not take 5 chips, Matthew will turn the table on her and leave her with 96 chips! So she must take 5 on her first turn in order to win.

9. A problem from the *Liber Abaci*, a math text written by Fibonnaci in the 13th century:

> On a certain ground there are two towers, one of which is 30 feet high, the other 40, and they are only 50 feet apart; two birds descending together from the heights of the two towers fly to the center of a fountain between the towers; the distance from the center [of the fountain] to the foot of the higher tower is sought.

In this problem assume: the birds are flying at the same speed, depart at the same time, and arrive together at the fountain; and the fountain and feet of the towers are collinear.

(A) 18 **(B)** 20 **(C)** 22 **(D)** 24 **(E)** 32

Solution: A Let x = the distance from the center of the fountain to the base of the higher tower. Then $50 - x$ = the distance from the center of the fountain to the base of the lower tower. By the Pythagorean Theorem $x^2 + 40^2 = (50 - x)^2 + 30^2$ or $x^2 + 1600 = x^2 - 100x + 2500 + 900$ or $100x = 1800$ or $x = 18$.

10. The trapezoid $ABCD$ has $AB \parallel CD$, $AB = 5$, and $DC = 12$. Draw EF parallel to AB with E on AD and F on BC. If EF splits trapezoid $ABCD$ into two trapezoids of equal area, what is the length of EF?

(A) 9 **(B)** $\dfrac{120}{17}$ **(C)** $\dfrac{17}{2}$ **(D)** $\dfrac{13\sqrt{2}}{2}$ **(E)** $2\sqrt{15}$

Solution: D Let $x = EF$. Let h and k be the altitudes of trapezoids $EFCD$

and $ABFE$ respectively. The equality of the areas is equivalent to $(x+12)h=(x+5)k$ while $h/k=(12-x)/(x-5)$. From $(12-x)/(x-5)=(x+5)/(x+12)$ we obtain $x=13\sqrt{2}/2$, which happens to (always) be the root-mean-square average of the bases.

11. How many ordered pairs (x,y) of positive integers satisfy both

$$\frac{x}{8}+\frac{y}{3}>1 \qquad \text{and} \qquad \frac{x}{12}+\frac{y}{7}<1$$

(A) 22 **(B)** 23 **(C)** 24 **(D)** 25 **(E)** 26

Solution: E By Pick's Theorem, the area of the quadrilateral with vertices at $(8,0),(12,0),(0,7),(0,3)=B/2+I-1$ where B is the number of lattice points on the boundary and I is both the number of lattice points in the interior and the number of integer solutions to our system. Thus $(1/2)(12\cdot 7-3\cdot 8)=1/2(5+5)+I-1$ or $30=4+I$ or $I=26$.

12. A parallelogram has sides of length 7 and 9. Its longer diagonal has length 14. What is the length of its shorter diagonal?

(A) 8 **(B)** 8.5 **(C)** 9 **(D)** 9.5 **(E)** 10

Solution: A By the Parallelogram Law, $14^2+d^2=2(7^2+9^2)=260$ or $d^2=64$ or $d=8$, where d is the other diagonal, in this case shorter. Or one can use the Law of Cosines twice.

13. Find a number c so that the three distinct solutions $x_1<x_2<x_3$ of the equation $x^3+6x^2-8x+4=c$ satisfy $x_1+x_3=2x_2$.

(A) 36 **(B)** 37 **(C)** 38 **(D)** 39 **(E)** 40

Solution: A Since $x_1+x_2+x_3=-6=3x_2$, then $x_2=-2$ is a root. Then $(-2)^3+6(-2)^2-8(-2)+4=-8+24+16+4=36=c$. And in this case indeed all three roots are real and $x_2=-2$ is in the middle.

14. Point z_0 of the complex plane lies in the Mandelbrot set if and only if the set of points $\{z_0,z_1,z_2,\cdots\}$ lies inside some circle, where $z_{n+1}=z_n^2+z_0$. Which of the following points does **not** belong to the Mandelbrot set?

(A) 0 **(B)** $\frac{1}{2}$ **(C)** -1 **(D)** i **(E)** $-i$

Solution: B The other choices lead to cycles so they clearly are in the

Mandelbrot set. By elimination it must be choice (B). In fact it can be shown that if a is real and $a > 1/4$, then a does not belong to the Mandelbrot set.

15. If $f(x) = \dfrac{x}{x+1}$ find the value of the limit:

$$\lim_{h \to 0} \frac{f(2+4h) - 4f(2+3h) + 6f(2+2h) - 4f(2+h) + f(2)}{h^4}$$

(A) 0 **(B)** $-\dfrac{1}{10}$ **(C)** $-\dfrac{8}{81}$ **(D)** $\dfrac{1}{24}$ **(E)** does not exist

Solution: C This is forward fourth difference, meaning it is the fourth derivative of f at $x = 2$. Note that $f(x) = 1 - 1/(x+1)$ making it easy to find the fourth derivative of f. So $f^{(4)}(2) = -24/3^5 = -8/81$. Or use L'Hospital four times.

16. Let f be twice differentiable on the interval (a, b). Suppose $f > 0$ and $f'' > 0$ on (a, b). Then which of the following functions must be increasing on (a, b)?

I. f^2

II. $f \cdot f'$

III. $\dfrac{f'}{f}$

(A) I only **(B)** II only **(C)** I and II only **(D)** II and III only **(E)** I, II, and III

Solution: B Since $(f^2)' = 2f \cdot f'$, then f^2 is not necessarily increasing on (a, b) since $f' < 0$ can occur. Since $(f \cdot f')' = (f')^2 + f \cdot f'' > 0$, then $f \cdot f'$ is increasing on (a, b). But the numerator of $(f'/f)'$ is equal to $f'' \cdot f - (f')^2$ and there is no way to control its sign. So Roman III is not necessarily true.

17. If the point (a, b) on the curve $y = 8x - x^2$ is closest on this curve to the point $(-5, 19)$, find the value of $a + b$.

(A) 0 **(B)** 6 **(C)** 8 **(D)** 14 **(E)** 18

Solution: E The closest point is $(3, 15)$. Differentiate $(x+5)^2 + (y-19)^2$ with respect to x and set equal to zero: $2(x+5) + 2(y-17)dy/dx = 0$. Replace dy/dx by $8 - 2x$ and y by $8x - x^2$ to get $2x^3 - 24x^2 + 103x - 147 = 0$

whose only real root is $x = 3$. Then by the global first derivative test, also known as 'the only critical point in town test', we have that $x = 3$ corresponds to a global minimum.

18. Find the value of the limit

$$\lim_{x\to\infty}(\sqrt{x^2+2x+5}-x)^x$$

(A) 0 **(B)** 1 **(C)** e **(D)** $e^{3/2}$ **(E)** e^2

Solution: E After taking the ln use L'Hospitals Rule with patience and conjugates.

19. There exists a unique line $y = ax + b$ in the x, y coordinate plane which is tangent at two distinct points to the curve $y = x^4 - 8x^2 + 6x + 4$. Find the value of $a - b$.

(A) 18 **(B)** 21 **(C)** 22 **(D)** 26 **(E)** 29

Solution: A Since $x^4 - 8x^2 + 6x + 4 = (x^4 - 8x^2 + 16) + (6x - 12) = (x - 2)^2(x+2)^2 + (6x - 12)$ it follows that the line $y = 6x - 12$ is tangent to the curve at both $x = 2$ and $x = -2$. So $a - b = 6 + 12 = 18$.

20. Let $f(x) = (x + [[2x]])^{[[3x]]}$ for $x > 0$ where $[[x]]$ =greatest integer less than or equal to x. Find the value of $f'(0.7)$.

(A) 0 **(B)** 2.6 **(C)** 3.4 **(D)** 4.2 **(E)** does not exist

Solution: C Since for x near 0.7, $[[2x]] = 1$ and $[[3x]] = 2$, the function $g(x) = (x+1)^2$ will have the same derivative at 0.7 as $f(x)$ will. Then $g'(0.7) = 2(1.7) = 3.4$.