Fall 2012 McNabb GDCTM Contest
Pre-Algebra

NO Calculators Allowed

1. Which of these quantities is the greatest?

   (A) $\frac{3}{7}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{7}{15}$ **(D)** $\frac{11}{19}$ **(E)** $\frac{13}{27}$

2. How many millions are in a trillion?

   (A) 3 **(B)** 10^2 **(C)** 10^3 **(D)** 10^5 **(E)** 10^6

3. How many numbers are in the list

   $$21, 13, 5, -3, -11, \cdots, -203, -211$$

   where each number is 8 less than the one before it?

   (A) 20 **(B)** 23 **(C)** 28 **(D)** 29 **(E)** 30

4. If each of 32 boys and 32 girls receives 32 gifts then how many gifts in total were received?

   (A) 2^{10} **(B)** 2^{11} **(C)** 2^{12} **(D)** 2^{13} **(E)** 2^{15}

5. A gas tank went from 3/8 ths full to 2/3 rds full by adding seven gallons of gas. How many more gallons must now be added to completely fill the tank?

   (A) 6 **(B)** 7 **(C)** 8 **(D)** 11 **(E)** 24

6. What is the largest possible value of the greatest common factor of six different two-digit whole numbers?

   (A) 10 **(B)** 12 **(C)** 15 **(D)** 16 **(E)** 19

7. If $a = 5$ and $b = 3$, then the value of $4 - b(3 - a)$ is

   (A) -2 **(B)** 3 **(C)** 5 **(D)** 10 **(E)** 21

8. On Black Friday a store reduced its price on a camera by 30%. Two weeks later, the item still not having sold, the store reduced the Black Friday sale price by 50%. The final price on the camera is what per cent of its original price?

(A) 20 **(B)** 35 **(C)** 50 **(D)** 65 **(E)** 80

9. The smallest prime greater than 120 is equal to

(A) 127 **(B)** 129 **(C)** 131 **(D)** 133 **(E)** 137

10. In the sequence of numbers

$$a, b, 1, -1, 0, -1, -1, -2, \cdots$$

each number after the second is the sum of the previous two numbers. Find the value of a.

(A) -1 **(B)** 3 **(C)** 0 **(D)** 4 **(E)** 1

11. If $\frac{a}{b} = \frac{17}{4}$, $\frac{b}{c} = \frac{3}{7}$, $\frac{c}{d} = \frac{8}{17}$, and $\frac{d}{e} = \frac{7}{6}$, what is the value of $\frac{a}{e}$?

(A) $1/34$ **(B)** $1/2$ **(C)** 1 **(D)** 2 **(E)** 14

12. In how many ways can the letters in CHEETAH be arranged so that no two consecutive letters are the same?

(A) 660 **(B)** 540 **(C)** 1260 **(D)** 720 **(E)** 330

13. In Hezy's piggy bank, the value of all the pennies equals the value of all the nickels; the value of all the dimes is twice the value of all the nickels. If Hezy has only pennies, nickels, and dimes, and he has 210 coins total in his piggy bank, how much are all those coins worth?

(A) \$5.45 **(B)** \$6.00 **(C)** \$7.60 **(D)** \$8.00 **(E)** \$10.50

14. What is the smallest positive integer n that satisfies $17n - 31m = 1$ if m must also be a positive integer?

(A) 44 **(B)** 17 **(C)** 15 **(D)** 13 **(E)** 11

15. Sixty points are equally spaced entirely around a circle. How many regular polygons can be formed using these and only these points as vertices?

(A) 60 **(B)** 68 **(C)** 78 **(D)** 88 **(E)** 89

16. The integer 8027 is the product of exactly two primes. What is the sum of the digits of the larger of these two primes?

(A) 10 **(B)** 13 **(C)** 16 **(D)** 17 **(E)** 18

17. Cheryl and Matthew take turns removing chips from a pile of 101 chips. On each turn they must remove 1, 2, 3, 4, or 5 chips (which of these number of chips is up to them and can change or not from turn to turn). The winner is the person who removes the last chip or chips. If Cheryl goes first, how many chips should she remove to guarantee that she will win with best play, no matter how Matthew moves?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

18. A frog is on a number line and can jump either one unit to the left or one unit to the right. If it starts at the origin and jumps randomly 6 times, what is the probability it is back at the origin at the end of those 6 jumps?

(A) $\frac{1}{64}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{17}{32}$ **(E)** $\frac{5}{16}$

Fall 2012 McNabb GDCTM Contest
Algebra One

NO Calculators Allowed

1. On Black Friday a store reduced its price on a camera by 30%. Two weeks later, the item still not having sold, the store reduced the Black Friday sale price by 50%. The final price on the camera is what per cent of its original price?

   (A) 20 **(B)** 35 **(C)** 50 **(D)** 65 **(E)** 80

2. If one defines
$$(a, b) \wedge (c, d) = ad - bc$$
   solve this equation for x: $(2, x) \wedge (7, -4) = 3$

   (A) $-\dfrac{7}{11}$ **(B)** $\dfrac{11}{7}$ **(C)** $\dfrac{7}{11}$ **(D)** 11 **(E)** $-\dfrac{11}{7}$

3. In the sequence of numbers
$$a, b, 1, -1, 0, -1, -1, -2, \cdots$$
   each number after the second is the sum of the previous two numbers. Find the value of a.

   (A) -1 **(B)** 3 **(C)** 0 **(D)** 4 **(E)** 1

4. A certain triangle in the coordinate plane has area 6. Then the x coordinates of each vertex of this triangle are doubled, but the y coordinates of each vertex are left alone. What is the area of this new triangle?

   (A) 3 **(B)** 6 **(C)** 12 **(D)** 24 **(E)** cannot be determined

5. If $\frac{a}{b} = \frac{17}{4}$, $\frac{b}{c} = \frac{3}{7}$, $\frac{c}{d} = \frac{8}{17}$, and $\frac{d}{e} = \frac{7}{6}$, what is the value of $\frac{a}{e}$?

   (A) 1/34 **(B)** 1/2 **(C)** 1 **(D)** 2 **(E)** 14

6. In how many ways can the letters in CHEETAH be arranged so that no two consecutive letters are the same?

   (A) 660 **(B)** 540 **(C)** 1260 **(D)** 720 **(E)** 330

7. The points x, x^2, and x^3 are graphed on the number line below. Which could be the value of x?

(A) -2 **(B)** -1 **(C)** -1/2 **(D)** 1/3 **(E)** 2

8. What is the smallest positive integer n that satisfies $17n - 31m = 1$ if m must also be a positive integer?

(A) 44 **(B)** 17 **(C)** 15 **(D)** 13 **(E)** 11

9. In how many ways can 9 students be divided into 3 groups of 3 students each?

(A) 81 **(B)** 180 **(C)** 280 **(D)** 540 **(E)** 1680

10. How many solutions does the equation $|x - 2| = |4 - x|$ have?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** infinitely many

11. Which of the integers below can be expressed in the form $p^2 + q^2 + r^2 + s^2 + t^2$ where p, q, r, s and t are all odd integers?

(A) 2012 **(B)** 2013 **(C)** 2014 **(D)** 2015 **(E)** 2016

12. Sixty points are equally spaced entirely around a circle. How many regular polygons can be formed using these and only these points as vertices?

(A) 60 **(B)** 68 **(C)** 78 **(D)** 88 **(E)** 89

13. Cheryl and Matthew take turns removing chips from a pile of 101 chips. On each turn they must remove 1, 2, 3, 4, or 5 chips (which of these number of chips is up to them and can change or not from turn to turn). The winner is the person who removes the last chip or chips. If Cheryl goes first, how many chips should she remove to guarantee that she will win with best play, no matter how Matthew moves?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

14. A problem from the *Liber Abaci*, a math text written by Fibonnaci in the 13th century:

> On a certain ground there are two towers, one of which is 30 feet high, the other 40, and they are only 50 feet apart; two birds descending together from the heights of the two towers fly to the center of a fountain between the towers; the distance from the center [of the fountain] to the foot of the higher tower is sought.

In this problem assume: the birds are flying at the same speed, depart at the same time, and arrive together at the fountain; and the fountain and feet of the towers are collinear.

(A) 18 **(B)** 20 **(C)** 22 **(D)** 24 **(E)** 32

15. A boat goes downriver from A to B in 3 days and returns upriver from B to A in 4 days. How long in days would it take an inner tube to float downriver from A to B?

(A) 12 **(B)** 18 **(C)** 24 **(D)** 30 **(E)** 32

16. Find the value of x if

$$\begin{aligned} 3x + 2y - z &= 1 \\ -x + y - 3z &= 7 \\ x + 2y + 9z &= -1 \end{aligned}$$

(A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

17. A frog is on a number line and can jump either one unit to the left or one unit to the right. If it starts at the origin and jumps randomly 6 times, what is the probability it is back at the origin at the end of those 6 jumps?

(A) $\frac{1}{64}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{17}{32}$ **(E)** $\frac{5}{16}$

18. The coefficient of x^{18} in the product

$$(x+1)(x+3)(x+5)(x+7)\cdots(x+37)$$

is equal to

(A) 1 **(B)** 243 **(C)** 361 **(D)** 400 **(E)** 401

Fall 2012 McNabb GDCTM Contest
Geometry

NO Calculators Allowed

1. If one defines

$$(a, b) \wedge (c, d) = ad - bc$$

solve this equation for x: $(2, x) \wedge (7, -4) = 3$

(A) $-\frac{7}{11}$ **(B)** $\frac{11}{7}$ **(C)** $\frac{7}{11}$ **(D)** 11 **(E)** $-\frac{11}{7}$

2. A certain triangle in the coordinate plane has area 6. Then the x coordinates of each vertex of this triangle are doubled, but the y coordinates of each vertex are left alone. What is the area of this new triangle?

(A) 3 **(B)** 6 **(C)** 12 **(D)** 24 **(E)** cannot be determined

3. The points x, x^2, and x^3 are graphed on the number line below. Which could be the value of x?

(A) -2 **(B)** -1 **(C)** -1/2 **(D)** 1/3 **(E)** 2

4. In how many ways can the letters in CHEETAH be arranged so that no two consecutive letters are the same?

(A) 660 **(B)** 540 **(C)** 1260 **(D)** 720 **(E)** 330

5. What is the area of a rhombus with sides equal to 13 and short diagonal equal to 10?

(A) 60 **(B)** 65 **(C)** 120 **(D)** 130 **(E)** 260

6. In how many ways can 9 students be divided into 3 groups of 3 students each?

(A) 81 **(B)** 180 **(C)** 280 **(D)** 540 **(E)** 1680

7. In $\triangle ABC$, points D and E lie on sides AC and AB respectively. Draw BD and CE intersecting at point F. Suppose $AC = AB = 12$, $BF = FC = 6$, and $EF = FD = 5$. Find the length of AD.

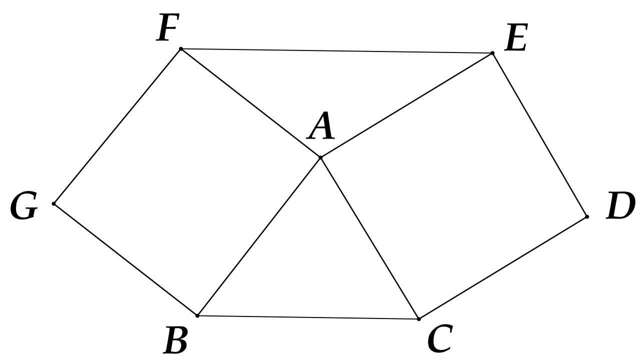
(A) 7 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

8. On the first test of the school year an algebra class averaged 81. If the three lowest scoring exams were not considered, the average would have been 84. If those three lowest scores were 52, 62, and 66, how many students are in the algebra class?

(A) 21 **(B)** 24 **(C)** 26 **(D)** 27 **(E)** 28

9. An equilateral triangle ABC fits between two squares, $AEDC$ and $ABGF$ as shown. Segment FE is drawn. What is the measure of $\angle AFE$ in degrees?

(A) 30 **(B)** 45 **(C)** 50 **(D)** 55 **(E)** 60



10. A problem from the *Liber Abaci*, a math text written by Fibonnaci in the 13th century:

> On a certain ground there are two towers, one of which is 30 feet high, the other 40, and they are only 50 feet apart; two birds descending together from the heights of the two towers fly to the center of a fountain between the towers; the distance from the center [of the fountain] to the foot of the higher tower is sought.

In this problem assume: the birds are flying at the same speed, depart at the same time, and arrive together at the fountain; and the fountain and feet of the towers are collinear.

(A) 18 **(B)** 20 **(C)** 22 **(D)** 24 **(E)** 32

11. Sixty points are equally spaced entirely around a circle. How many regular polygons can be formed using these and only these points as vertices?

(A) 60 **(B)** 68 **(C)** 78 **(D)** 88 **(E)** 89

12. There are two non-congruent triangles ABC with $AB = 8$, $BC = 5$, and $\angle A = 30^\circ$. What is the positive difference of their areas?

(A) 5 **(B)** 6 **(C)** $3\sqrt{3}$ **(D)** $5\sqrt{3}$ **(E)** 12

13. The coefficient of x^{18} in the product

$$(x+1)(x+3)(x+5)(x+7)\cdots(x+37)$$

is equal to

(A) 1 **(B)** 243 **(C)** 361 **(D)** 400 **(E)** 401

14. Let $a \neq b$. The equation of the perpendicular bisector of the segment with endpoints (a, b) and (b, a) is

(A) $y = x$ **(B)** $y = 0$ **(C)** $x = 0$ **(D)** $y = -x$ **(E)** $y = 2x$

15. Suppose that the statements:

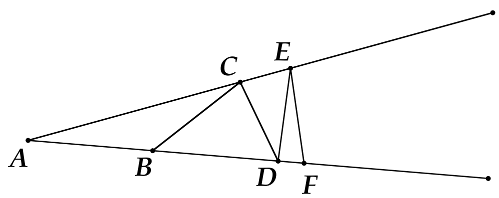
No *zoofs* are *zarns*
At least one *zune* is not a *zoof*

are true. Which of the following must be true?

(A) At least one *zune* is a *zoof*
(B) No *zarn* is a *zune*
(C) At least one *zarn* is not a *zune*
(D) All *zunes* are *zarns*
(E) None of the above

16. Points B, C, D, E, and F lie as shown on $\angle A$ with $AB = BC = CD = DE = EF$ as shown. If $\angle AEF$ is right, then find, in degrees, the measure of $\angle CAB$.

(A) 16 **(B)** 18 **(C)** 20 **(D)** 22 **(E)** 24



17. The line $4x - 8y = 15$ is irritating to graph as it contains no points of the form (a, b), with both a and b integers. Such points are called *lattice points*. What is the minimum distance between this line and the set of lattice points?

(A) $\frac{1}{12}$ **(B)** $\frac{1}{10}$ **(C)** $\frac{\sqrt{5}}{20}$
(D) $\frac{\sqrt{5}}{15}$ **(E)** no minimum exists

18. The real number $\sqrt{41 - 24\sqrt{2}}$ can be put in the form $a\sqrt{2} - b$ where a and b are positive integers. What is the value of $a + b$?

(A) 4 **(B)** 7 **(C)** 8 **(D)** 16 **(E)** 18

19. An ordered triple of positive integers (a, b, c) with $a < b < c$ and $a^2 + b^2 = c^2$ is called *Pythagorean*. Find the perimeter of the only Pythagorean triple with $a = 11$.

(A) 88 **(B)** 99 **(C)** 121 **(D)** 132 **(E)** 154

20. The trapezoid $ABCD$ has $AB \parallel CD$, $AB = 5$, and $DC = 12$. Draw EF parallel to AB with E on AD and F on BC. If EF splits trapezoid $ABCD$ into two trapezoids of equal area, what is the length of EF?

(A) 9 **(B)** $\dfrac{120}{17}$ **(C)** $\dfrac{17}{2}$ **(D)** $\dfrac{13\sqrt{2}}{2}$ **(E)** $2\sqrt{15}$

Fall 2012 McNabb GDCTM Contest
Algebra Two

NO Calculators Allowed

1. How many numbers are in the list

$$21, 13, 5, -3, -11, \cdots, -203, -211$$

where each number is 8 less than the one before it?

(A) 20 **(B)** 23 **(C)** 28 **(D)** 29 **(E)** 30

2. What is the largest possible value of the greatest common factor of six different two-digit whole numbers?

(A) 10 **(B)** 12 **(C)** 15 **(D)** 16 **(E)** 19

3. In the sequence of numbers

$$a, b, 1, -1, 0, -1, -1, -2, \cdots$$

each number after the second is the sum of the previous two numbers. Find the value of a.

(A) -1 **(B)** 3 **(C)** 0 **(D)** 4 **(E)** 1

4. A certain triangle in the coordinate plane has area 6. Then the x coordinates of each vertex of this triangle are doubled, but the y coordinates of each vertex are left alone. What is the area of this new triangle?

(A) 3 **(B)** 6 **(C)** 12 **(D)** 24 **(E)** cannot be determined

5. The points x, x^2, and x^3 are graphed on the number line below. Which could be the value of x?

(A) -2 **(B)** -1 **(C)** -1/2 **(D)** 1/3 **(E)** 2

6. There are two non-congruent triangles ABC with $AB = 8$, $BC = 5$, and $\angle A = 30^\circ$. What is the positive difference of their areas?

(A) 5 **(B)** 6 **(C)** $3\sqrt{3}$ **(D)** $5\sqrt{3}$ **(E)** 12

7. In how many ways can the letters in CHEETAH be arranged so that no two consecutive letters are the same?

(A) 660 **(B)** 540 **(C)** 1260 **(D)** 720 **(E)** 330

8. The coefficient of x^{18} in the product

$$(x+1)(x+3)(x+5)(x+7)\cdots(x+37)$$

is equal to

(A) 1 **(B)** 243 **(C)** 361 **(D)** 400 **(E)** 401

9. On the first test of the school year an algebra class averaged 81. If the three lowest scoring exams were not considered, the average would have been 84. If those three lowest scores were 52, 62, and 66, how many students are in the algebra class?

(A) 21 **(B)** 24 **(C)** 26 **(D)** 27 **(E)** 28

10. In $\triangle ABC$, points D and E lie on sides AC and AB respectively. Draw BD and CE intersecting at point F. Suppose $AC = AB = 12$, $BF = FC = 6$, and $EF = FD = 5$. Find the length of AD.

(A) 7 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

11. If the point $(1, -5)$ lies on the graph of $y = -f(1-2x) + 2$ which point below must lie on the graph of $y = 3f(5x-6) - 8$?

(A) $(1, 13)$ **(B)** $(3, -8)$ **(C)** $(-1, -8)$ **(D)** $(0, 11)$ **(E)** $(-4, 8)$

12. How many real solutions are there to the equation

$$(x+1)(x+2)(x+3)(x+4) = (x+5)(x+6)(x+7)(x+8)$$

?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

13. How many ordered pairs (x, y) of positive integers satisfy both

$$\frac{x}{8} + \frac{y}{3} > 1 \qquad \text{and} \qquad \frac{x}{12} + \frac{y}{7} < 1$$

(A) 22 **(B)** 23 **(C)** 24 **(D)** 25 **(E)** 26

14. A boat goes downriver from A to B in 3 days and returns upriver from B to A in 4 days. How long in days would it take an inner tube to float downriver from A to B?

(A) 12 **(B)** 18 **(C)** 24 **(D)** 30 **(E)** 32

15. Which of the following are true for all positive real numbers a and b?

I. $\sqrt{ab} = \sqrt{a}\sqrt{b}$

II. $\sqrt{a} + \sqrt{b} > \sqrt{a+b}$

III. $\sqrt{\dfrac{a^2+b^2}{2}} < \dfrac{a+b}{2}$

(A) I only **(B)** I and II only **(C)** I and III only
(D) II and III only **(E)** I, II, and III

16. Three congruent circles are mutually externally tangent to each other and each circle is tangent to a different side of an equilateral triangle of side length 2. The common radius of the circles can be written in the form $a - \sqrt{b}$, where a and b are positive integers. What is the value of $3a + b$?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

17. Find a number c so that the three distinct solutions $x_1 < x_2 < x_3$ of the equation $x^3 + 6x^2 - 8x + 4 = c$ satisfy $x_1 + x_3 = 2x_2$.

(A) 36 **(B)** 37 **(C)** 38 **(D)** 39 **(E)** 40

18. Find the value of x if

$$\begin{aligned} 3x + 2y - z &= 1 \\ -x + y - 3z &= 7 \\ x + 2y + 9z &= -1 \end{aligned}$$

(A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

19. At how many points do the graphs of $y = x^4 + x^3 - 2x^2 + x$ and $y = x$ intersect?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

20. A caravan of 5 cars is headed down I-35 for the annual Texas A&M High School Math Tournament. In how many ways can the caravan be re-formed after a rest stop so that each non-leading car has a different car in front of it from the one it had before the rest stop? Note the car that lead before the rest stop may still be leading after it and also could be any of the following cars after it as well.

(A) 31 **(B)** 53 **(C)** 60 **(D)** 65 **(E)** 71

Fall 2012 McNabb GDCTM Contest
PreCalculus

NO Calculators Allowed

1. If 10% of a is b what is 10% of b?

   (A) $100a$ **(B)** $10a$ **(C)** a **(D)** $.1a$ **(E)** $.01a$

2. If 10 carpenters can build 10 cabinets in 10 days how many days does it take 20 carpenters to build 20 cabinets?

   (A) 5 **(B)** 10 **(C)** 15 **(D)** 20 **(E)** 25

3. Express the fraction

$$\frac{1}{1+\frac{1}{3+\frac{1}{5}}}$$

   in lowest terms.

   (A) **(B)** **(C)** **(D)** 16/21 **(E)**

4. The square root of 20000 lies between

   (A) **(B)** 140 and 141 **(C)** 141 and 142 **(D)** **(E)** 10,000 and 10,001

5. The last 6 digits of 13^{426} are 000009. What is the sum of the last 6 digits of 13^{1704}?

   (A) 18 **(B)** **(C)** **(D)** **(E)**

6. In the repeating decimal $0.\overline{71771}$, in which decimal place does the 2013th 7 appear?

   (A) 671 **(B)** 2014 **(C)** 2015 **(D)** 3354 **(E)** 3355

7. How many positive factors does 2013 have?

(A) **(B)** 8 **(C)** **(D)** **(E)**

8. If the integer $\underline{4400b074}$ is divisible by 101, what must the digit b equal?

(A) 0 **(B)** 2 **(C)** 3 **(D)** 5 **(E)** 8

9. Using estimation, the number of digits in the number 2^{50} is

(A) between 6 and 10 inclusive **(B)** between 11 and 15 inclusive **(C)** between 16 and 20 inclusive
(D) between 21 and 25 inclusive **(E)** 26 or more

10. For how many positive integers n does $n!$ end in exactly eleven zero's?

(A) 0 **(B)** 3 **(C)** 5 **(D)** 8 **(E)** 11

11. Four roses and two tulips are to be arranged in a circle. Two such arrangements are considered to be the same if and only if each can be rotated into the other. How many distinct arrangements are possible?

(A) 3 **(B)** **(C)** **(D)** **(E)**

12. How many solutions in radians of $\sin 2\theta = \cos 3\theta$ lie in the interval $[0, 2\pi]$?

(A) 0 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 6

13. The product of a certain integer and 180 is a perfect square. That certain integer must be divisible by

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

14. When a cyclist gets a puncture she has just completed three-fourths of her route. She finishes her route by walking. If she spent twice as much time walking as biking , how many times faster does she bike than walk?

(A) 4 **(B)** 4.5 **(C)** 5 **(D)** 5.5 **(E)** 6

15. Each face of a cube is numbered with a positive integer in such a way that the numbers on pairs of faces sharing an edge differ by at least two. What is the minimum possible sum of six such integers?

(A) 12 **(B)** 15 **(C)** 18 **(D)** 24 **(E)** 27

16. In the following arrangement of the positive integers, in which column, counting from left to right, does 7021 appear?

$$\begin{array}{cccc} & & & \cdot \\ & & 5 & \cdots \\ & 2 & 6 & \cdots \\ 1 & 3 & 7 & \cdots \\ & 4 & 8 & \cdots \\ & & 9 & \cdots \\ & & & \cdot \end{array}$$

(A) 43 **(B)** 51 **(C)** 52 **(D)** 84 **(E)** 99

17. The sum of a set of numbers is the sum of all the numbers in that set. How many subsets of the set $\{1,2,3,4,5,6,7\}$ have a sum of 12?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

18. Solar system X has three planets A, B, and C which orbit uniformly in concentric circles about a single star S at the center of those circles in such a way that planet A completes exactly 8 orbits in one Earth year, planet B exactly 4 orbits in one Earth year, and planet C exactly 2 orbits in one Earth year. When a satellite from Earth first observes Solar system X it records that S, A, B, and C all lie on the same line. In the course of one full Earth year, how many times will the satellite observe this phenomenon? Count the original and final observations.

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

19. A purse may only contain pennies, nickels, dimes, and quarters but does not have to contain any particular type of coin, except as demanded in meeting the following conditions: the average value of the coins in the

purse is 16 cents; if one more quarter were added to it the average value would rise to 17 cents. How many quarters are actually in the purse?

(A) 0 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** cannot be uniquely determined

20. In a dart game, each throw of the dart yields either 3 points or b points, where b is a positive integer. If there are ten unattainable positive total scores, then the value of b could be

(A) 4 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 12

21. How many ordered triples (a, b, c) of positive integers satisfy $a + b + 3c = 17$?

(A) **(B)** **(C)** **(D)** 35 **(E)**

22. Which transformation never changes the median of a list of a dozen distinct positive integers?

(A) adding 6 to each number in the list
(B) adding 3 to each of the three smallest numbers in the list
(C) subtracting 4 from each of the four largest numbers in the list
(D) doubling each number in the list
(E) taking the reciprocal of each number in the list

23. A careless librarian has reshelved the 5 volumes of an art encyclopedia in the correct order. Each volume has its binding facing out, which is correct of course, but has a 1/4 probability of being upside down. What is the probability that exactly one pair of front covers are now face to face?

(A) **(B)** **(C)** **(D)** **(E)** 69/128

24. Twenty-seven unit cubes are assembled to form a $3 \times 3 \times 3$ cube. If two of the unit cubes are then chosen at random, what is the probability they share a face?

(A) **(B)** **(C)** **(D)** **(E)** 2/13

25. The value of

$$1+2+3+4-5+6+7+8+9-10+\cdots+46+47+48+49-50$$

is equal to

(A) **(B)** **(C)** 700 **(D)** **(E)**

26. Exchanging the positions of two numbers in a list (and nothing else) is called a *swap*. What is the minimum number of swaps needed to put the list

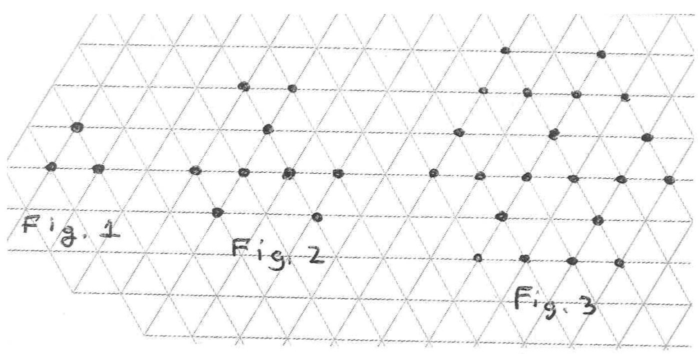
$$9,7,3,1,10,8,5,4,6,2$$

in increasing order?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

27. The first three figures of a certain sequence of figures are shown below on an equilateral triangle grid. Each successor figure is obtained recursively from its predeccesor by this rule: any two or more consecutive dots on a grid line generate new neighboring dots on that grid line, on either side, where no dot was before. All previous dots remain. How many dots does the 5th figure in this sequence have?

(A) 34 **(B)** 36 **(C)** 57 **(D)** 59 **(E)** 64



28. How many positive integers less than 100,000 have no two consecutive digits odd?

(A) **(B)** **(C)** **(D)** 8124 **(E)**

29. Two sides of a parallelogram lie along the lines $x-y+1=0$ and $2x+3y-6=0$. If the diagonals of the parallelogram meet at the point $(1,1/2)$, find the area of this parallelogram.

(A) **(B)** **(C)** **(D)** 3 **(E)**

30. Amanda and Blake together can paint a room in 7 hours. Blake and Cathy together paint it in 5 hours. Cathy and Amanda together paint it in 6 hours. How long would it take in hours to paint the room if all three work together?

(A) 420/107 **(B)** 210/53 **(C)** 4 **(D)** 6 **(E)** 105/23

31. Given the three points $(2013, -1863)$, $(1776, -1812)$, and $(1181, -1492)$ in the coordinate plane, a fourth point (a, b) is called a *complementing* point if it along with the given three points form the vertices of a parallelogram. Find the sum of all the coordinates of all the complementing points of the given three points.

(A) -197 **(B)** **(C)** **(D)** **(E)**

32. Which of the following equations has exactly two solutions over the real numbers?

(A) $x^2 - 6x + 9 = 0$ **(B)** $5x = 2(5 - 7x)$ **(C)** $|x + 8| = -5$
(D) $|x| = 12$ **(E)** $x^2 + 1 = 0$

33. An amount of 10000 dollars is deposited in an account for one year at an interest rate of x percent per year compounded twice a year. If at the end of the year 10404 is in the account, then x is

(A) 3.9 **(B)** 4 **(C)** 4.1 **(D)** 7.8 **(E)** 8

34. Suppose a and b are given real numbers with $a > b > 0$. If the triangle formed by the lines $y = ax$, $y = bx$, and $x = 1$ has area 2013, what is the value of $a - b$?

(A) 2012 **(B)** 2013 **(C)** 3013 **(D)** 4024 **(E)** 4026

35. Let a, b, and n be constants, with n a positive integer. If the first three terms of the binomial expansion of $(a + x)^n$ are, in ascending powers of x, equal to $3b + 6bx + 5bx^2$, then find the value of $a + b + n$.

(A) **(B)** **(C)** **(D)** **(E)** 252

36. In pentagon $ABCDE$, $AB = AE = 3$, $BC = DE = 1$, $CD = 3$, $\angle B = \angle E$, and $\angle A$ is right. The area of this pentagon lies between

(A) 6 and 7 **(B)** 7 and 8 **(C)** 8 and 9
(D) 9 and 10 **(E)** 10 and 11

37. An off-center balance does balance when pan A has a weight of 600 grams while pan B has a weight of 900 grams. If a weight of 400 grams is added to pan A, how many grams must be added to pan B to restore the balance? Neglect the mass of the pans, beams, etc...

(A) 400 **(B)** 500 **(C)** 600 **(D)** 700 **(E)** 900

38. In $\triangle ABC$, points D and E lie on sides AB and AC respectively and DE is parallel to BC. If $BC = 10$, $BD = 2$, and $AE = 3$, then AD *cannot* be equal to

(A) $2/3$ **(B)** $5/2$ **(C)** $7/2$ **(D)** $9/2$ **(E)** $21/2$

39. Quadrilateral $PQRS$ is inscribed in a circle. Segments PQ and SR are extended to meet at T. If $\angle SPQ = 80°$ and $\angle PQR = 130°$, find in degrees the measure of $\angle T$.

(A) 50 **(B)** **(C)** **(D)** **(E)**

40. A circle of radius 9 is externally tangent to a second circle of radius b. If a common tangent to the two circles has length 12, what is the value of b?

(A) 3.5 **(B)** 4 **(C)** 6 **(D)** 7.5 **(E)** 9

41. Three coplanar and mutually parallel lines l, m, and n are situated so that m lies between the other two, the distance from l to m is 1 and the distance from m to n is 2. Points A, B, and C lie on lines l, m, and n respectively and $\triangle ABC$ is equilateral. What is the area of $\triangle ABC$?

(A) **(B)** **(C)** **(D)** **(E)** $\frac{7\sqrt{3}}{3}$

42. A solid opaque cube of side length 5 meters rests on flat ground. It is illuminated only by powerful point-source of light located 5 meters above one of the cube's top corners. Find the area in square meters of the shadow cast by the cube on the ground.

(A) 75 **(B)** 85 **(C)** $50\sqrt{2}$ **(D)** 91 **(E)** 100

43. How many different paths are there from $(0,0)$ to $(4,4)$ if only these three kinds of steps may be taken: (i) a unit step to the right, (ii) a unit step up, (iii) a northeast diagonal step from point (i,j) to point $(i+1,j+1)$?

(A) **(B)** **(C)** 321 **(D)** **(E)**

44. The set of points in space equidistant from two skew lines is

(A) the empty set **(B)** a single point **(C)** a line
(D) the union of two intersecting lines **(E)** none of the above

45. In triangle ABC, the angle bisector CD of $\angle A$ has point D on side AB. If $AC = 1$, $BC = \sqrt{3}$, $AD = \sqrt{3}-1$ and $DB = 3-\sqrt{3}$, then what is the length CD?

(A) **(B)** $\sqrt{6-3\sqrt{3}}$ **(C)** **(D)** **(E)**

46. Three concentric circles have radii of length 2, 5, and $\sqrt{80}$ respectively. What is the maximum possible area of a triangle having a vertex on each of these circles?

(A) 33 **(B)** $10\sqrt{10}$ **(C)** 34 **(D)** $15\sqrt{5}$ **(E)** $20\sqrt{3}$

47. The integral

$$\int_0^{\pi/2} \frac{1}{1+\cos\theta}\, d\theta$$

have value

(A) **(B)** 1 **(C)** **(D)** **(E)** diverges

48. Find the minimum possible value of the expression $6\cos x + 2\cos 2x + 5$.

(A) **(B)** $3/4$ **(C)** **(D)** **(E)**

49. One factor of $14x^2 + 37x + 24$ is

(A) **(B)** **(C)** **(D)** $7x + 8$ **(E)**

50. Find the sum

$$\sum_{n=1}^{\infty} \frac{4n+6}{n^4 + 6n^3 + 11n^2 + 6n}$$

(A) $4/3$ **(B)** **(C)** **(D)** **(E)**

51. For how many positive real values of the constant k is the following statement true: $\int_0^k (2k-2)x^k \, dx = 81$.

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

52. Determine

$$\lim_{n\to\infty} \int_0^{\pi/6} \sin^n x \, dx$$

(A) 0 **(B)** **(C)** **(D)** **(E)** does not exist

53. The improper integral $\int_0^{\infty} \frac{1}{1+e^x} \, dx$ has the value

(A) $\ln 2$ **(B)** $1/2$ **(C)** $2/3$ **(D)** e **(E)** does not converge

54. A thin rod lies along the x-axis with endpoints at $x = 2$ and $x = 8$. If the density of the rod at each point is directly proportional to the point's distance to the origin, what is the x-coordinate of the center of mass of the rod?

(A) **(B)** **(C)** **(D)** **(E)**

55. Given that $\int_0^{10} \ln(x^2 - 10x + 26) \, dx = k$ then find the value of $\int_0^{10} x \ln(x^2 - 10x + 26) \, dx$.

(A) 0 **(B)** k **(C)** $2k$ **(D)** $k \ln 2$ **(E)** $5k$

Fall 2012 McNabb GDCTM Contest
Calculus

NO Calculators Allowed

Assume all variables are real unless otherwise stated in the problem.

1. If $\frac{a}{b} = \frac{17}{4}$, $\frac{b}{c} = \frac{3}{7}$, $\frac{c}{d} = \frac{8}{17}$, and $\frac{d}{e} = \frac{7}{6}$, what is the value of $\frac{a}{e}$?

   (A) 1/34 **(B)** 1/2 **(C)** 1 **(D)** 2 **(E)** 14

2. In how many ways can the letters in CHEETAH be arranged so that no two consecutive letters are the same?

   (A) 660 **(B)** 540 **(C)** 1260 **(D)** 720 **(E)** 330

3. The coefficient of x^{18} in the product

$$(x+1)(x+3)(x+5)(x+7)\cdots(x+37)$$

   is equal to

   (A) 1 **(B)** 243 **(C)** 361 **(D)** 400 **(E)** 401

4. What is the smallest positive integer n that satisfies $17n - 31m = 1$ if m must also be a positive integer?

   (A) 44 **(B)** 17 **(C)** 15 **(D)** 13 **(E)** 11

5. A boat goes downriver from A to B in 3 days and returns upriver from B to A in 4 days. How long in days would it take an inner tube to float downriver from A to B?

   (A) 12 **(B)** 18 **(C)** 24 **(D)** 30 **(E)** 32

6. On the first test of the school year an algebra class averaged 81. If the three lowest scoring exams were not considered, the average would have been 84. If those three lowest scores were 52, 62, and 66, how many students are in the algebra class?

   (A) 21 **(B)** 24 **(C)** 26 **(D)** 27 **(E)** 28

7. Suppose that the statements:

No *zoofs* are *zarns*
At least one *zune* is not a *zoof*

are true. Which of the following must be true?

(A) At least one *zune* is a *zoof*
(B) No *zarn* is a *zune*
(C) At least one *zarn* is not a *zune*
(D) All *zunes* are *zarns*
(E) None of the above

8. Cheryl and Matthew take turns removing chips from a pile of 101 chips. On each turn they must remove 1, 2, 3, 4, or 5 chips (which of these number of chips is up to them and can change or not from turn to turn). The winner is the person who removes the last chip or chips. If Cheryl goes first, how many chips should she remove to guarantee that she will win with best play, no matter how Matthew moves?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

9. A problem from the *Liber Abaci*, a math text written by Fibonnaci in the 13th century:

> On a certain ground there are two towers, one of which is 30 feet high, the other 40, and they are only 50 feet apart; two birds descending together from the heights of the two towers fly to the center of a fountain between the towers; the distance from the center [of the fountain] to the foot of the higher tower is sought.

In this problem assume: the birds are flying at the same speed, depart at the same time, and arrive together at the fountain; and the fountain and feet of the towers are collinear.

(A) 18 **(B)** 20 **(C)** 22 **(D)** 24 **(E)** 32

10. The trapezoid $ABCD$ has $AB \parallel CD$, $AB = 5$, and $DC = 12$. Draw EF parallel to AB with E on AD and F on BC. If EF splits trapezoid $ABCD$ into two trapezoids of equal area, what is the length of EF?

(A) 9 **(B)** $\frac{120}{17}$ **(C)** $\frac{17}{2}$ **(D)** $\frac{13\sqrt{2}}{2}$ **(E)** $2\sqrt{15}$

11. How many ordered pairs (x, y) of positive integers satisfy both

$$\frac{x}{8} + \frac{y}{3} > 1 \qquad \text{and} \qquad \frac{x}{12} + \frac{y}{7} < 1$$

(A) 22 **(B)** 23 **(C)** 24 **(D)** 25 **(E)** 26

12. A parallelogram has sides of length 7 and 9. Its longer diagonal has length 14. What is the length of its shorter diagonal?

(A) 8 **(B)** 8.5 **(C)** 9 **(D)** 9.5 **(E)** 10

13. Find a number c so that the three distinct solutions $x_1 < x_2 < x_3$ of the equation $x^3 + 6x^2 - 8x + 4 = c$ satisfy $x_1 + x_3 = 2x_2$.

(A) 36 **(B)** 37 **(C)** 38 **(D)** 39 **(E)** 40

14. Point z_0 of the complex plane lies in the Mandelbrot set if and only if the set of complex points $\{z_0, z_1, z_2, \cdots\}$ lies inside some circle, where $z_{n+1} = z_n^2 + z_0$. Which of the following points does **not** belong to the Mandelbrot set?

(A) 0 **(B)** $\frac{1}{2}$ **(C)** -1 **(D)** i **(E)** $-i$

15. If $f(x) = \dfrac{x}{x+1}$ find the value of the limit:

$$\lim_{h \to 0} \frac{f(2+4h) - 4f(2+3h) + 6f(2+2h) - 4f(2+h) + f(2)}{h^4}$$

(A) 0 **(B)** $-\frac{1}{10}$ **(C)** $-\frac{8}{81}$ **(D)** $\frac{1}{24}$ **(E)** does not exist

16. Let f be twice differentiable on the interval (a, b). Suppose $f > 0$ and $f'' > 0$ on (a, b). Then which of the following functions must be increasing on (a, b)?

I. f^2

II. $f \cdot f'$

III. $\frac{f'}{f}$

(A) I only **(B)** II only **(C)** I and II only **(D)** II and III only **(E)** I, II, and III

17. If the point (a, b) on the curve $y = 8x - x^2$ is closest on this curve to the point $(-5, 19)$, find the value of $a + b$.

(A) 0 **(B)** 6 **(C)** 7 **(D)** 13 **(E)** 18

18. Find the value of the limit

$$\lim_{x\to\infty} (\sqrt{x^2 + 2x + 5} - x)^x$$

(A) 0 **(B)** 1 **(C)** e **(D)** $e^{3/2}$ **(E)** e^2

19. There exists a unique line $y = ax + b$ in the x, y coordinate plane which is tangent at two distinct points to the curve $y = x^4 - 8x^2 + 6x + 4$. Find the value of $a - b$.

(A) 18 **(B)** 21 **(C)** 22 **(D)** 26 **(E)** 29

20. Let $f(x) = (x + [[2x]])^{[[3x]]}$ for $x > 0$ where $[[x]]$ =greatest integer less than or equal to x. Find the value of $f'(0.7)$.

(A) 0 **(B)** 2.6 **(C)** 3.4 **(D)** 4.2 **(E)** does not exist