Fall 2012 McNabb GDCTM Contest
Calculus

NO Calculators Allowed

Assume all variables are real unless otherwise stated in the problem.

1. If $\frac{a}{b} = \frac{17}{4}$, $\frac{b}{c} = \frac{3}{7}$, $\frac{c}{d} = \frac{8}{17}$, and $\frac{d}{e} = \frac{7}{6}$, what is the value of $\frac{a}{e}$?

   (A) 1/34 **(B)** 1/2 **(C)** 1 **(D)** 2 **(E)** 14

2. In how many ways can the letters in CHEETAH be arranged so that no two consecutive letters are the same?

   (A) 660 **(B)** 540 **(C)** 1260 **(D)** 720 **(E)** 330

3. The coefficient of x^{18} in the product

$$(x+1)(x+3)(x+5)(x+7)\cdots(x+37)$$

   is equal to

   (A) 1 **(B)** 243 **(C)** 361 **(D)** 400 **(E)** 401

4. What is the smallest positive integer n that satisfies $17n - 31m = 1$ if m must also be a positive integer?

   (A) 44 **(B)** 17 **(C)** 15 **(D)** 13 **(E)** 11

5. A boat goes downriver from A to B in 3 days and returns upriver from B to A in 4 days. How long in days would it take an inner tube to float downriver from A to B?

   (A) 12 **(B)** 18 **(C)** 24 **(D)** 30 **(E)** 32

6. On the first test of the school year an algebra class averaged 81. If the three lowest scoring exams were not considered, the average would have been 84. If those three lowest scores were 52, 62, and 66, how many students are in the algebra class?

   (A) 21 **(B)** 24 **(C)** 26 **(D)** 27 **(E)** 28

7. Suppose that the statements:

No *zoofs* are *zarns*
At least one *zune* is not a *zoof*

are true. Which of the following must be true?

(A) At least one *zune* is a *zoof*
(B) No *zarn* is a *zune*
(C) At least one *zarn* is not a *zune*
(D) All *zunes* are *zarns*
(E) None of the above

8. Cheryl and Matthew take turns removing chips from a pile of 101 chips. On each turn they must remove 1, 2, 3, 4, or 5 chips (which of these number of chips is up to them and can change or not from turn to turn). The winner is the person who removes the last chip or chips. If Cheryl goes first, how many chips should she remove to guarantee that she will win with best play, no matter how Matthew moves?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

9. A problem from the *Liber Abaci*, a math text written by Fibonnaci in the 13th century:

> On a certain ground there are two towers, one of which is 30 feet high, the other 40, and they are only 50 feet apart; two birds descending together from the heights of the two towers fly to the center of a fountain between the towers; the distance from the center [of the fountain] to the foot of the higher tower is sought.

In this problem assume: the birds are flying at the same speed, depart at the same time, and arrive together at the fountain; and the fountain and feet of the towers are collinear.

(A) 18 **(B)** 20 **(C)** 22 **(D)** 24 **(E)** 32

10. The trapezoid $ABCD$ has $AB \parallel CD$, $AB = 5$, and $DC = 12$. Draw EF parallel to AB with E on AD and F on BC. If EF splits trapezoid $ABCD$ into two trapezoids of equal area, what is the length of EF?

(A) 9 **(B)** $\dfrac{120}{17}$ **(C)** $\dfrac{17}{2}$ **(D)** $\dfrac{13\sqrt{2}}{2}$ **(E)** $2\sqrt{15}$

11. How many ordered pairs (x, y) of positive integers satisfy both

$$\frac{x}{8} + \frac{y}{3} > 1 \qquad \text{and} \qquad \frac{x}{12} + \frac{y}{7} < 1$$

(A) 22 **(B)** 23 **(C)** 24 **(D)** 25 **(E)** 26

12. A parallelogram has sides of length 7 and 9. Its longer diagonal has length 14. What is the length of its shorter diagonal?

(A) 8 **(B)** 8.5 **(C)** 9 **(D)** 9.5 **(E)** 10

13. Find a number c so that the three distinct solutions $x_1 < x_2 < x_3$ of the equation $x^3 + 6x^2 - 8x + 4 = c$ satisfy $x_1 + x_3 = 2x_2$.

(A) 36 **(B)** 37 **(C)** 38 **(D)** 39 **(E)** 40

14. Point z_0 of the complex plane lies in the Mandelbrot set if and only if the set of complex points $\{z_0, z_1, z_2, \cdots\}$ lies inside some circle, where $z_{n+1} = z_n^2 + z_0$. Which of the following points does **not** belong to the Mandelbrot set?

(A) 0 **(B)** $\dfrac{1}{2}$ **(C)** -1 **(D)** i **(E)** $-i$

15. If $f(x) = \dfrac{x}{x+1}$ find the value of the limit:

$$\lim_{h \to 0} \frac{f(2+4h) - 4f(2+3h) + 6f(2+2h) - 4f(2+h) + f(2)}{h^4}$$

(A) 0 **(B)** $-\dfrac{1}{10}$ **(C)** $-\dfrac{8}{81}$ **(D)** $\dfrac{1}{24}$ **(E)** does not exist

16. Let f be twice differentiable on the interval (a, b). Suppose $f > 0$ and $f'' > 0$ on (a, b). Then which of the following functions must be increasing on (a, b)?

I. f^2

II. $f \cdot f'$

III. $\dfrac{f'}{f}$

(A) I only **(B)** II only **(C)** I and II only **(D)** II and III only **(E)** I, II, and III

17. If the point (a, b) on the curve $y = 8x - x^2$ is closest on this curve to the point $(-5, 19)$, find the value of $a + b$.

(A) 0 **(B)** 6 **(C)** 7 **(D)** 13 **(E)** 18

18. Find the value of the limit

$$\lim_{x \to \infty} (\sqrt{x^2 + 2x + 5} - x)^x$$

(A) 0 **(B)** 1 **(C)** e **(D)** $e^{3/2}$ **(E)** e^2

19. There exists a unique line $y = ax + b$ in the x, y coordinate plane which is tangent at two distinct points to the curve $y = x^4 - 8x^2 + 6x + 4$. Find the value of $a - b$.

(A) 18 **(B)** 21 **(C)** 22 **(D)** 26 **(E)** 29

20. Let $f(x) = (x + [[2x]])^{[[3x]]}$ for $x > 0$ where $[[x]] =$greatest integer less than or equal to x. Find the value of $f'(0.7)$.

(A) 0 **(B)** 2.6 **(C)** 3.4 **(D)** 4.2 **(E)** does not exist