Fall 2012 McNabb GDCTM Contest
PreCalculus

NO Calculators Allowed

1. If 10% of a is b what is 10% of b?

   (A) $100a$ **(B)** $10a$ **(C)** a **(D)** $.1a$ **(E)** $.01a$

2. If 10 carpenters can build 10 cabinets in 10 days how many days does it take 20 carpenters to build 20 cabinets?

   (A) 5 **(B)** 10 **(C)** 15 **(D)** 20 **(E)** 25

3. Express the fraction

$$\frac{1}{1+\dfrac{1}{3+\dfrac{1}{5}}}$$

   in lowest terms.

   (A) **(B)** **(C)** **(D)** 16/21 **(E)**

4. The square root of 20000 lies between

   (A) **(B)** 140 and 141 **(C)** 141 and 142 **(D)** **(E)** 10,000 and 10,001

5. The last 6 digits of 13^{426} are 000009. What is the sum of the last 6 digits of 13^{1704}?

   (A) 18 **(B)** **(C)** **(D)** **(E)**

6. In the repeating decimal $0.\overline{71771}$, in which decimal place does the 2013th 7 appear?

   (A) 671 **(B)** 2014 **(C)** 2015 **(D)** 3354 **(E)** 3355

7. How many positive factors does 2013 have?

   (A) **(B)** 8 **(C)** **(D)** **(E)**

8. If the integer $\underline{4400b074}$ is divisible by 101, what must the digit b equal?

   (A) 0 **(B)** 2 **(C)** 3 **(D)** 5 **(E)** 8

9. Using estimation, the number of digits in the number 2^{50} is

   (A) between 6 and 10 inclusive **(B)** between 11 and 15 inclusive **(C)** between 16 and 20 inclusive
   (D) between 21 and 25 inclusive **(E)** 26 or more

10. For how many positive integers n does $n!$ end in exactly eleven zero's?

    (A) 0 **(B)** 3 **(C)** 5 **(D)** 8 **(E)** 11

11. Four roses and two tulips are to be arranged in a circle. Two such arrangements are considered to be the same if and only if each can be rotated into the other. How many distinct arrangements are possible?

    (A) 3 **(B)** **(C)** **(D)** **(E)**

12. How many solutions in radians of $\sin 2\theta = \cos 3\theta$ lie in the interval $[0, 2\pi]$?

    (A) 0 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 6

13. The product of a certain integer and 180 is a perfect square. That certain integer must be divisible by

    (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

14. When a cyclist gets a puncture she has just completed three-fourths of her route. She finishes her route by walking. If she spent twice as much time walking as biking , how many times faster does she bike than walk?

    (A) 4 **(B)** 4.5 **(C)** 5 **(D)** 5.5 **(E)** 6

15. Each face of a cube is numbered with a positive integer in such a way that the numbers on pairs of faces sharing an edge differ by at least two. What is the minimum possible sum of six such integers?

(A) 12 **(B)** 15 **(C)** 18 **(D)** 24 **(E)** 27

16. In the following arrangement of the positive integers, in which column, counting from left to right, does 7021 appear?

$$\begin{array}{cccc} & & & \cdot \\ & & 5 & \cdots \\ & 2 & 6 & \cdots \\ 1 & 3 & 7 & \cdots \\ & 4 & 8 & \cdots \\ & & 9 & \cdots \\ & & & \cdot \end{array}$$

(A) 43 **(B)** 51 **(C)** 52 **(D)** 84 **(E)** 99

17. The sum of a set of numbers is the sum of all the numbers in that set. How many subsets of the set $\{1, 2, 3, 4, 5, 6, 7\}$ have a sum of 12?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

18. Solar system X has three planets A, B, and C which orbit uniformly in concentric circles about a single star S at the center of those circles in such a way that planet A completes exactly 8 orbits in one Earth year, planet B exactly 4 orbits in one Earth year, and planet C exactly 2 orbits in one Earth year. When a satellite from Earth first observes Solar system X it records that S, A, B, and C all lie on the same line. In the course of one full Earth year, how many times will the satellite observe this phenomenon? Count the original and final observations.

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

19. A purse may only contain pennies, nickels, dimes, and quarters but does not have to contain any particular type of coin, except as demanded in meeting the following conditions: the average value of the coins in the

purse is 16 cents; if one more quarter were added to it the average value would rise to 17 cents. How many quarters are actually in the purse?

(A) 0 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** cannot be uniquely determined

20. In a dart game, each throw of the dart yields either 3 points or b points, where b is a positive integer. If there are ten unattainable positive total scores, then the value of b could be

(A) 4 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 12

21. How many ordered triples (a, b, c) of positive integers satisfy $a + b + 3c = 17$?

(A) **(B)** **(C)** **(D)** 35 **(E)**

22. Which transformation never changes the median of a list of a dozen distinct positive integers?

(A) adding 6 to each number in the list
(B) adding 3 to each of the three smallest numbers in the list
(C) subtracting 4 from each of the four largest numbers in the list
(D) doubling each number in the list
(E) taking the reciprocal of each number in the list

23. A careless librarian has reshelved the 5 volumes of an art encyclopedia in the correct order. Each volume has its binding facing out, which is correct of course, but has a $1/4$ probability of being upside down. What is the probability that exactly one pair of front covers are now face to face?

(A) **(B)** **(C)** **(D)** **(E)** $69/128$

24. Twenty-seven unit cubes are assembled to form a $3 \times 3 \times 3$ cube. If two of the unit cubes are then chosen at random, what is the probability they share a face?

(A) **(B)** **(C)** **(D)** **(E)** $2/13$

25. The value of

$$1 + 2 + 3 + 4 - 5 + 6 + 7 + 8 + 9 - 10 + \cdots + 46 + 47 + 48 + 49 - 50$$

is equal to

(A) **(B)** **(C)** 700 **(D)** **(E)**

26. Exchanging the positions of two numbers in a list (and nothing else) is called a *swap*. What is the minimum number of swaps needed to put the list

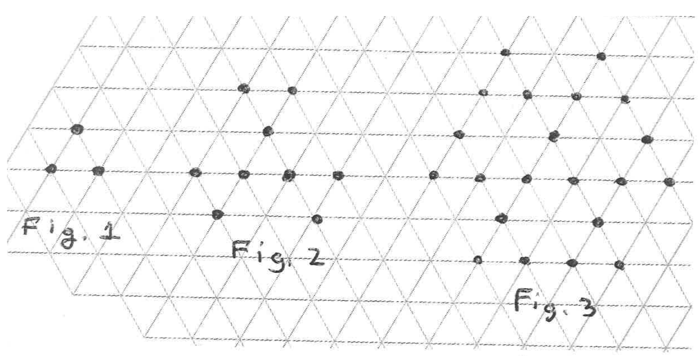
$$9, 7, 3, 1, 10, 8, 5, 4, 6, 2$$

in increasing order?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

27. The first three figures of a certain sequence of figures are shown below on an equilateral triangle grid. Each successor figure is obtained recursively from its predeccesor by this rule: any two or more consecutive dots on a grid line generate new neighboring dots on that grid line, on either side, where no dot was before. All previous dots remain. How many dots does the 5th figure in this sequence have?

(A) 34 **(B)** 36 **(C)** 57 **(D)** 59 **(E)** 64



28. How many positive integers less than 100,000 have no two consecutive digits odd?

(A) **(B)** **(C)** **(D)** 8124 **(E)**

29. Two sides of a parallelogram lie along the lines $x - y + 1 = 0$ and $2x + 3y - 6 = 0$. If the diagonals of the parallelogram meet at the point $(1, 1/2)$, find the area of this parallelogram.

(A) **(B)** **(C)** **(D)** 3 **(E)**

30. Amanda and Blake together can paint a room in 7 hours. Blake and Cathy together paint it in 5 hours. Cathy and Amanda together paint it in 6 hours. How long would it take in hours to paint the room if all three work together?

(A) 420/107 **(B)** 210/53 **(C)** 4 **(D)** 6 **(E)** 105/23

31. Given the three points $(2013, -1863)$, $(1776, -1812)$, and $(1181, -1492)$ in the coordinate plane, a fourth point (a, b) is called a *complementing* point if it along with the given three points form the vertices of a parallelogram. Find the sum of all the coordinates of all the complementing points of the given three points.

(A) -197 **(B)** **(C)** **(D)** **(E)**

32. Which of the following equations has exactly two solutions over the real numbers?

(A) $x^2 - 6x + 9 = 0$ **(B)** $5x = 2(5 - 7x)$ **(C)** $|x + 8| = -5$
(D) $|x| = 12$ **(E)** $x^2 + 1 = 0$

33. An amount of 10000 dollars is deposited in an account for one year at an interest rate of x percent per year compounded twice a year. If at the end of the year 10404 is in the account, then x is

(A) 3.9 **(B)** 4 **(C)** 4.1 **(D)** 7.8 **(E)** 8

34. Suppose a and b are given real numbers with $a > b > 0$. If the triangle formed by the lines $y = ax$, $y = bx$, and $x = 1$ has area 2013, what is the value of $a - b$?

(A) 2012 **(B)** 2013 **(C)** 3013 **(D)** 4024 **(E)** 4026

35. Let a, b, and n be constants, with n a positive integer. If the first three terms of the binomial expansion of $(a + x)^n$ are, in ascending powers of x, equal to $3b + 6bx + 5bx^2$, then find the value of $a + b + n$.

(A) **(B)** **(C)** **(D)** **(E)** 252

36. In pentagon $ABCDE$, $AB = AE = 3$, $BC = DE = 1$, $CD = 3$, $\angle B = \angle E$, and $\angle A$ is right. The area of this pentagon lies between

(A) 6 and 7 **(B)** 7 and 8 **(C)** 8 and 9
(D) 9 and 10 **(E)** 10 and 11

37. An off-center balance does balance when pan A has a weight of 600 grams while pan B has a weight of 900 grams. If a weight of 400 grams is added to pan A, how many grams must be added to pan B to restore the balance? Neglect the mass of the pans, beams, etc...

(A) 400 **(B)** 500 **(C)** 600 **(D)** 700 **(E)** 900

38. In $\triangle ABC$, points D and E lie on sides AB and AC respectively and DE is parallel to BC. If $BC = 10$, $BD = 2$, and $AE = 3$, then AD *cannot* be equal to

(A) $2/3$ **(B)** $5/2$ **(C)** $7/2$ **(D)** $9/2$ **(E)** $21/2$

39. Quadrilateral $PQRS$ is inscribed in a circle. Segments PQ and SR are extended to meet at T. If $\angle SPQ = 80°$ and $\angle PQR = 130°$, find in degrees the measure of $\angle T$.

(A) 50 **(B)** **(C)** **(D)** **(E)**

40. A circle of radius 9 is externally tangent to a second circle of radius b. If a common tangent to the two circles has length 12, what is the value of b?

(A) 3.5 **(B)** 4 **(C)** 6 **(D)** 7.5 **(E)** 9

41. Three coplanar and mutually parallel lines l, m, and n are situated so that m lies between the other two, the distance from l to m is 1 and the distance from m to n is 2. Points A, B, and C lie on lines l, m, and n respectively and $\triangle ABC$ is equilateral. What is the area of $\triangle ABC$?

(A) **(B)** **(C)** **(D)** **(E)** $\frac{7\sqrt{3}}{3}$

42. A solid opaque cube of side length 5 meters rests on flat ground. It is illuminated only by powerful point-source of light located 5 meters above one of the cube's top corners. Find the area in square meters of the shadow cast by the cube on the ground.

(A) 75 **(B)** 85 **(C)** $50\sqrt{2}$ **(D)** 91 **(E)** 100

43. How many different paths are there from $(0,0)$ to $(4,4)$ if only these three kinds of steps may be taken: (i) a unit step to the right, (ii) a unit step up, (iii) a northeast diagonal step from point (i,j) to point $(i+1,j+1)$?

(A) **(B)** **(C)** 321 **(D)** **(E)**

44. The set of points in space equidistant from two skew lines is

(A) the empty set **(B)** a single point **(C)** a line
(D) the union of two intersecting lines **(E)** none of the above

45. In triangle ABC, the angle bisector CD of $\angle A$ has point D on side AB. If $AC = 1$, $BC = \sqrt{3}$, $AD = \sqrt{3} - 1$ and $DB = 3 - \sqrt{3}$, then what is the length CD?

(A) **(B)** $\sqrt{6 - 3\sqrt{3}}$ **(C)** **(D)** **(E)**

46. Three concentric circles have radii of length 2, 5, and $\sqrt{80}$ respectively. What is the maximum possible area of a triangle having a vertex on each of these circles?

(A) 33 **(B)** $10\sqrt{10}$ **(C)** 34 **(D)** $15\sqrt{5}$ **(E)** $20\sqrt{3}$

47. The integral

$$\int_0^{\pi/2} \frac{1}{1+\cos\theta}\, d\theta$$

have value

(A) **(B)** 1 **(C)** **(D)** **(E)** diverges

48. Find the minimum possible value of the expression $6\cos x + 2\cos 2x + 5$.

(A) **(B)** $3/4$ **(C)** **(D)** **(E)**

49. One factor of $14x^2 + 37x + 24$ is

(A) **(B)** **(C)** **(D)** $7x + 8$ **(E)**

50. Find the sum

$$\sum_{n=1}^{\infty} \frac{4n+6}{n^4 + 6n^3 + 11n^2 + 6n}$$

(A) 4/3 **(B)** **(C)** **(D)** **(E)**

51. For how many positive real values of the constant k is the following statement true: $\int_0^k (2k-2)x^k \, dx = 81$.

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

52. Determine

$$\lim_{n \to \infty} \int_0^{\pi/6} \sin^n x \, dx$$

(A) 0 **(B)** **(C)** **(D)** **(E)** does not exist

53. The improper integral $\int_0^\infty \frac{1}{1+e^x} \, dx$ has the value

(A) $\ln 2$ **(B)** 1/2 **(C)** 2/3 **(D)** e **(E)** does not converge

54. A thin rod lies along the x-axis with endpoints at $x = 2$ and $x = 8$. If the density of the rod at each point is directly proportional to the point's distance to the origin, what is the x-coordinate of the center of mass of the rod?

(A) **(B)** **(C)** **(D)** **(E)**

55. Given that $\int_0^{10} \ln(x^2 - 10x + 26) \, dx = k$ then find the value of $\int_0^{10} x \ln(x^2 - 10x + 26) \, dx$.

(A) 0 **(B)** k **(C)** $2k$ **(D)** $k \ln 2$ **(E)** $5k$