Spring 2011 McNabb GDCTM Contest Solutions

NO Calculators Allowed

1. Hezy leaves home for work at 6:45am. He drives to the Green Line train station 3 miles away at an average speed of 30 mph. After 8 minutes he boards the train for downtown. The train averages 45 mph for its 9 mile journey. After a 7 minute walk Hezy arrives at work. What time does Hezy arrive at work?

   Solution: It takes Hezy one-tenth of an hour or 6 minutes to get to the train station. The train itself takes a fifth of an hour or 12 minutes to get downtown. So it takes Hezy $6 + 8 + 12 + 7 = 33$ minutes to make his journey. Thus he arrives at $\mathbf{7:18}$.

2. How many arrangements of $REVERE$ are there in which the first R occurs before the first E?

   Solution: If the first R occurs in the first slot, there are $5!/3! = 20$ ways to arrange the other letters. If the first R before the first E occurs in the second slot, then the first letter must be the V and there are 4 ways to arrange the remaining letters. Thus the answer is $\mathbf{20 + 4 = 24}$.

3. In a class, $2/3$ of the students have brown eyes and $4/5$ of the students have brown hair. If students with brown eyes are twice as likely to have brown hair as students who do not have brown eyes, what fraction of the class has neither brown eyes nor brown hair? In a class of 150 students, let 96 have brown eyes and brown hair, 24 have not brown eyes and brown hair, 4 have brown eyes and not brown hair, and 26 have not brown eyes and brown hair. This meets the set-up of the problem. So the answer is $\mathbf{26/150 = 13/75}$

4. If today is a Saturday, what day of the week will it be 1001 days from today?

   Solution: Since 1001 is divisible by 7 it will be a **Saturday** again.

5. The sum of all the factors of 1001 is equal to

   Solution: Since $1001 = 7 \cdot 11 \cdot 13$ the sum of its factors is $(1 + 7)(1 + 11)(1 + 13) = \mathbf{1344}$.

6. When three different numbers from the set $\{-7, -2, -1, 0, 4, 5\}$ are multiplied together the smallest possible product is

   Solution: It is $(-\mathbf{7})(\mathbf{4})(\mathbf{5}) = -\mathbf{140}$.

7. Out of a sphere of clay with diameter 12, Marty fashions two spheres of radius 3 and 5 respectively. Using all of the remaining clay Jennifer fashions a sphere. What is the diameter of Jennifer's sphere?

   Solution: Since $6^3 = 3^3 + 5^3 + 4^3$, then Jennifer's sphere's radius is 4 so the diameter is **8**.

8. Let m and n be integers satisfying $m^2 + n^2 = 50$. The value of $m + n$ must be
   Solution: There are a number of different values. From $7^2 + 1^2 = 50$ we get 8 while from, among others, $5^2 + 5^2 = 25$ we get 10. So **cannot be uniquely determined**

9. Suppose that \$600 is divided into two parts in the ratio of 2 : 3. The larger of these parts is then further subdivided into two parts in the ratio of 3 : 2. The smallest of these now three parts is

   Solution: Since $600(3/5)(3/5) = 216$ and $600(3/5)(2/5) = 144$ the three parts are $144, 216, 240$. The smallest is **144**.

10. Points A, B, and C all lie on the same straight line and occur in the order given. If $AB/BC = 2/5$ and $AC = 35$, what is AB?

    Solution: $AB/AC = 2/7$ so $AB = (2/7)(35) = \mathbf{10}$.

11. Suppose the integer n has prime factorization $2^6 \cdot 3^8$. How many perfect square factors does n have?

    Solution: $2^a \cdot 3^b$ is a perfect square factor of n iff $a = 0, 2, 4, 6$ and $b = 0, 2, 4, 6, 8$. So there are $4 \cdot 5 = \mathbf{20}$ such.

12. The product $60 \times 60 \times 24 \times 7$ equals

    Solution: **The number of seconds in one week.**

13. The four digit integer $1A8B$ is divisible by 77. What is the value of $A + B$?

    Solution: It must be divisible by 11 so $1 - A + 8 - B$ is divisible by 11 so **A+B=9** is the only possibility since A and B are digits.

14. The surface area of sphere A is 50. Its volume is $\frac{1}{27}$th of the volume of sphere B. What is the surface area of sphere B?

    Solution: So the linear scale factor in moving from A to B is 3. So area changes by a factor of $3^2 = 9$. Thus the surface area of B is $9 \cdot 50 = \mathbf{450}$.

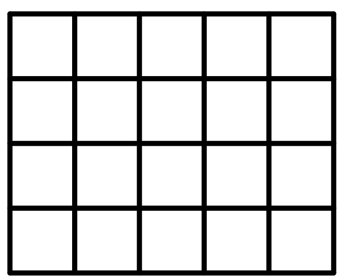
15. If n and m are positive integers and $480n = m^2$, what is the smallest possible value of m?

    Solution: The prime factorization of 480 is $2^5 \cdot 3 \cdot 5$. To make a perfect square all the exponents in the prime factorization must be even. Thus the least m^2 is $2^6 \cdot 3^2 \cdot 5^2$ so $\mathbf{m = 2^3 \cdot 3 \cdot 5 = 120}$.

16. A fifth number n is added to the set $\{3, 6, 9, 10\}$ to form a new set $\{3, 6, 9, 10, n\}$. For how many values of n is the mean of this new set equal to its own median?

    Solution: The possible medians are 6, n, and 9. Each is possible with exactly one correct value of n. So there are **3** such n.

17. How many rectangles are in this figure?



    Solution: There are 30 corners to choose first. After this there are 20 corners to choose not on the same horizontal or vertical line as the first corner. Choosing these two corners in this manner overcounts the number of rectangle by a factor of 4. So there are $30(20)/4 = \mathbf{150}$ such rectangles.

18. A regular 52 card deck is well shuffled. What is the probability that both the top and bottom cards are aces?

    Solution: $P = 4 \cdot 50! \cdot 3/52! = \mathbf{1/221}$

19. In two years a son will be one-third as old as his father was 2 years ago. In eighteen years this son will be the same age as his father was 18 years ago.

How old is the son now? Solution: Trial and Error or Guess and Correct. Try the middle answer of 14. 14+18=32 and 32+18=50 so the father would be 50 now. And $16 \cdot 3 = 50 - 2$ so **14** is correct!

20. How many different rectangular prisms can be made using exactly 48 unit cubes? Two prisms are the same if one can be rotated to coincide with the other. For example, a $3 \times 4 \times 4$ rectangular prism is the same as a $4 \times 3 \times 4$ rectangular prism.

    Solution: The **nine** ways are $(1,1,48)$, $(1,2,24)$, $(1,3,16)$, $(1,4,12)$, $(1,6,8)$, $(2,2,12)$, $(2,3,8)$, $(2,4,6)$, and $(3,4,4)$.

21. Recall that a Pythagorean triple is a triple (a,b,c) of positive integers satisfying $a^2 + b^2 = c^2$. Which of the following must be true?

    (I.) At least one of a, b, and c must be odd

    (II.) At least one of a, b, and c must be even

    (III.) For at least one Pythagorean triple, $a = b$.

    Solution. I is false since $(4,6,10)$ is a triple. II. is true since if two of sides are odd the third must be even. III is false since $\sqrt{2}$ is irrational. So **onlyII**.

22. The twelve edges of a cube are marked with the integers 1 through 12 in such a way that each edge receives a different number. Then each vertex of the cube is assigned the number equal to the sum of the numbers on the edges that meet at that vertex. Finally each face of the cube is assigned the sum of the numbers at each vertex of that face. What must be the sum of all the numbers that have been assigned to the faces of the cube?

    Solution: Note $1 + 2 + \cdots + 12 = 78$. Each of these gets used twice in forming the vertex sum. Then each gets used more times in forming the faces sum. So the face sum is $78 \cdot 2 \cdot 3 = \mathbf{468}$.

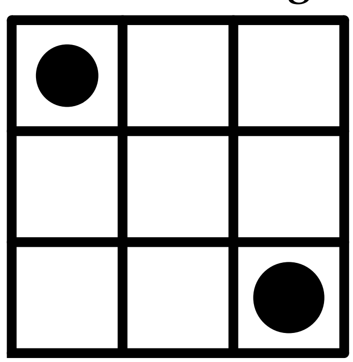
23. Councilman Bob sits on a 12 member City Council. A committee of 4 councilpeople is to be selected at random. What is probability that Bob is selected to be on this committee?

    Solution: $P = \frac{\binom{11}{3}}{\binom{12}{4}} = \mathbf{4}/9$.

24. In a class of 28 students, 20 take Latin, 14 take Greek, and 10 take Hebrew. If no student takes all three languages and 6 take no language, how many students must be taking both Greek and Hebrew? Solution: Exactly **2** must take both Greek and Hebrew. Note at most 2 can take a language other than Latin since 20 take Latin, and 22 take at least one language. Also students in both Greek and Hebrew cannot take Latin as no student takes all three. If fewer than 2 take both Greek and Hebrew then fewer than 22 take any language, a contradiction. The answer is **2**.

25. Molly's Motel is adopting a new room key system. The new keys will be square 3×3 cards each with two holes punched in them as in the figure. The two sides (what we would have called the front and back except we cannot tell which is which!) of such a card cannot be distinguished but there is a distinguished edge which is the edge to be inserted in the lock. What is the greatest number of rooms Molly's Motel can have?

insert this edge



Solution. 6 choices of the holes are symmetric with respect to the central line perpendicular to the edge to be inserted. So these count with full weight of one. The other 30 are not so symmetric and thus each has a pair (its reflection across this line) which is really the same. So these have only half-weigth. Thus there are at most $\mathbf{6 + 30/2 = 21}$ rooms.

26. If $a \blacktriangle b = b(a+1)$ what is the value of $(a \blacktriangle 1) \blacktriangle (1 \blacktriangle a)$? Solution: $(a \blacktriangle 1) \blacktriangle (1 \blacktriangle a) = (a+1) \blacktriangle (2a) = (2a)(a+2) = 2a^2 + 4a$

27. The sum of two positive numbers is S and their positive difference is $1/m$th of the smaller number. What is the value of the larger number?

Solution: Solve the system $x + y = S$ and $x - y = y/m$ where x is the larger number, to get $\mathbf{x = (m+1)S/(2m+1)}$.

28. Let a, b, x, and $y > 0$. If $x = by$ and $y = ax$ find the value of $\frac{a}{1+a} + \frac{b}{1+b}$. Solution: Note $b = 1/a$, so the desired expression becomes $\frac{a(1+b)+b(1+a)}{(1+a)(1+b)} = \frac{a+b+2ab}{a+b+ab+1} = \frac{\mathbf{a+b+2}}{\mathbf{a+b+2}} = \mathbf{1}$.

29. In two years a son will be one-third as old as his father was 2 years ago. In eighteen years this son will be the same age as his father was 18 years ago. How old is the son now?

    Solution: Solve the system $x + 2 = (1/3)(y-2)$ and $x + 18 = y - 18$ to get $\mathbf{x = 14}$.

30. Let $f(x)$ be a linear function satisfying $f(0) = 0$. If both $f(a+b) = 7$ and $f(a-2b) = 3$, then the value of $f(a+7b)$ must be

    Solution: $f(x) = mx$ which justifies: $f(a+7b) = f(3(a+b) - 2(a-2b)) = 3f(a+b) - 2f(a-2b) = 3(7) - 2(3) = \mathbf{15}$.

31. A train having to journey x miles in h hours, ran for k hours at a rate of r miles per hour, then stopped for m minutes. How fast must it go (in mph) on the rest of its journey to arrive on time?

    Solution: Let s denote the unknown rate. Then $h = k + m/60 + (x - kr)/s$. Solve for s to get $\mathbf{s} = \frac{\mathbf{60(x-kr)}}{\mathbf{60h - 60k - m}}$

32. The image of the line $y = 4x - 6$ under reflection across the line $y = -x$ is the line

    Solution: Map (x, y) to $(-y, -x)$. so the new line is given by $-x = 4(-y) - 6$, or $\mathbf{y = (1/4)x - 3/2}$.

33. The area of rectangle $ABCD$ is 40. Point P is on AB so that $BP = 3$. Point R is on AD so that $DR = 2$. Given that $APQR$ is a rectangle with area of 15, find the average of the two possible values for the length of AP.

    Solution: Let $x = AP$ so $AR = 15/x$. From $(x+3)(15/x+2) = 40$ we get $2x^2 - 19x + 45 = (x-5)(2x-9) = 0$. The average of 5 and 9/2 is $\mathbf{19/4}$.

34. What is the remainder when $x^{14} + x^{11} + x^8 + x^5 + x^3 + x^2 + x + 1$ is divided by $x^2 - x + 1$?

Solution: $x^2 - x + 1$ is a factor of $x^3 + 1$ so is always a factor of $x^{n+3} + x^n$. This leaves $x^2 + x$. Dividing $x^2 - x + 1$ into $x^3 + 1$ leaves a remainder of $2x - 1$.

35. In an isosceles trapezoid with bases 6 and 16 a circle is inscribed (touching all four sides of the trapezoid). What is the diameter of this circle?

Solution: By the equality of tangents to a circle, the non-parallel sides of the trapezoid have length 11. So if d is the diameter, $d^2 + 5^2 = 11^2$ which leads to $\mathbf{d} = \sqrt{\mathbf{96}} = \mathbf{4}\sqrt{\mathbf{6}}$

36. In the coordinate plane a triangle has vertices at $(0,0)$, $(28,0)$, and $(8,15)$. The circle inscribed in this triangle has center at (a,b). What is the value of $a + b$?

Solution: Use mass points. Weight each vertex proportional to the length of the side opposite it. So the coordinates of the incenter equal

$$\frac{25(0,0) + 28(8,15) + 17(28,0)}{25 + 28 + 17} = (10,6)$$

so the answer is $\mathbf{10} + \mathbf{6} = \mathbf{16}$

37. In $\triangle ABC$, point D is on segment $\overline{AC}$ and point E is on segment $\overline{AB}$. Segments $\overline{BD}$ and $\overline{CE}$ are drawn, intersecting at piont F. If $AE/EB = 2/5$ and $EF/FC = 3/7$ then the ratio BF/FD is equal to

Solution: Use mass points. Put a mass of 5 at A, of 2 at B, and of 3 at C. This leads to a mass of 2 at B balancing a mass of 8 at D. Thus $BF/FD = 8/2 = \mathbf{4}$

38. Triangle ABC is inscribed in a circle and $AB = AC = 6$. Point D lies on BC with $AD = 4$. AD is extended through D to E on the circle. Find DE. Solution: Note that $\triangle ABD \sim \triangle AEB$, so $AB^2 = AD \cdot AE$. Thus $6^2 = 4(4 + x)$ or $\mathbf{x} = \mathbf{5}$.

39. The series

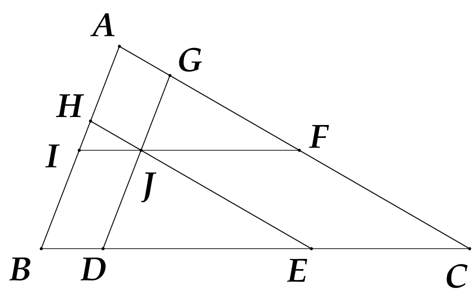
$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + 100 \cdot 101$$

has the value Solution: By using finite differences or looking at small cases show that $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1) = (1/3)(n^3 + 3n^2 + 2n) = (1/3)(n)(n+1)(n+2)$. So the answer is $\mathbf{100}(\mathbf{101})(\mathbf{102})/\mathbf{3} = \mathbf{343400}$.

40. Ten coplanar lines are given such that (i) exactly three lines are parallel (each to the other two), (ii) no other pairs of lines are parallel, and (iii) no three lines are concurrent. How many triangles are formed by these ten lines?

    Solution: The 7 non-parallel lines from $\binom{7}{3}$ triangles. Using two lines from the non-parallel lines with one of the three parallel lines gives $3 \cdot \binom{7}{2}$ triangles. Altogether there are then $\mathbf{35} + \mathbf{63} = \mathbf{98}$ triangles.

41. In triangle ABC the transversals DG, EH, and FI are concurrent at J, with $DG \parallel AB$, $EH \parallel AC$, and $FI \parallel BC$. If these three transversals have the same length, what is their common length if it is known that $AB = 8$, $BC = 16$, and $CA = 12$? Solution: As usual let $a = BC$, $b = AC$, and $c = AB$. Also

    let $\alpha = BE$, $\beta = CG$, and $\gamma = AI$. Let $L = DG = EH = FI$. From similar triangles, $\beta = \dfrac{b}{a}(a - \alpha) + \dfrac{b}{c}(c - \gamma)$ and $L = a\gamma/c = b\alpha/a = c\beta/b$. Thus $\beta = ab\gamma/c^2 = 2b - b\alpha/a - b\gamma/c$. Dividing through by b and collecting c we obtain $(a + c)\gamma/c^2 = 2 - \alpha/a$. Replace γ by $cb\alpha/a^2$, add α/a to both sides and make common denominators to yield $\alpha(ab + bc + ca)/(a^2c) = 2$ so $\alpha = 2a^2c/(ab + bc + ca)$. But $L = b\alpha/a$. Thus multiplying the expression for α by b/a on both sides gives

    $$L = \frac{2abc}{ab + bc + ca}$$

    Plugging in $a = 16$, $b = 12$, and $c = 8$ gives $L = \mathbf{96/13}$.

42. When $(a - b + c)^7$ is expanded and simplified how many terms are preceded by a minus sign?

Solution: We must collect the odd powers of b. Terms with b^1: 7 of them. Terms with b^3: 5 of them. Terms with b^5: 3 of them. Terms with b^7: 1 of them. So the answer is $\mathbf{7+5+3+1=16}$

43. The coefficient of x^8 when $(1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8)^3$ is expanded and similar terms are collected is equal to Solution: It is the same as the number of ordered solutions in non-negative integers to $a+b+c=8$. Use stars and bars (2 bars!) to get $\binom{\mathbf{10}}{\mathbf{2}} = \mathbf{45}$ ways.

44. Determine the number of ordered pairs (x, y) satisfying the system

$$2x^2 - xy - 6y^2 = 0$$
$$3x^2 - 5xy - 2y^2 = x + 5y + 2$$

Solution: Factor these equations as $(x-2y)(2x+3y) = 0$ and $(x-2y-1)(3x+y+2) = 0$. This gives three lines but two of them are parallel. So there are only **3** solutions.

45. Let each of a, b, x, and y be greater than one. If $\log_{ab} x = b$ and $\log_{ab} y = a$, then what is the value of $\log_{xy}(ab)$? Solution: $x = (ab)^a$, $y = (ab)^a$, so $xy = (ab)^{a+b}$ and $ab = (xy)^{1/(a+b)}$. Hence the answer is $\mathbf{1/(a+b)}$.

46. If $r+s=1$ and $r^4+s^4=4$ find the largest possible value of r^2+s^2.

Solution: Let $r^2+s^2=\alpha$. Then $(r+s)^4=1$ leads to $2rs(2r^2+3rs+2s^2)=-3$. But $2rs = 1-\alpha$ and we get $(1-\alpha)(\alpha+3) = -6$, or $\alpha^2+2\alpha-9=0$. Thus $\alpha = -1 \pm \sqrt{10}$ so the largest value of α is $\mathbf{-1+\sqrt{10}}$.

47. Let w and z be complex conjugate numbers such that w^2/z is a real number. If $|w-z| = 2\sqrt{2}$, what is the value of $|w|^2$?

Solution: Let $w = a+bi$ and $z = a-bi$. So $|w-z| = |2bi| = 2b = 2\sqrt{2}$, so $b = \sqrt{2}$. Note w^3 must be real. So then $3a^2bi - b^3i = 0$, or $3\sqrt{2}a^2 - 2\sqrt{2} = 0$, so $a^2 = 2/3$. Thus $\mathbf{a^2+b^2=2+2/3=8/3}$

48. In $\triangle ABC$, $AB = \sqrt{2011}$, $\angle C = 120^\circ$, and sides CA and CB are integers. The value of $CA+CB$ could be

Solution: $2011 = n^2+mn+m^2 = 39^2+39\cdot10+10^2$. So it could be $\mathbf{39+10=49}$

49. The polynomial $p(x) = x^4 - 5x^2 - 6x - 5$ has exactly two real roots, which occur in the form $\dfrac{A \pm \sqrt{B}}{2}$ where A and B are positive integers. Find the value of $A + B$.
Solution: $p(x) = x(x^3 - 1) - 5(x^2 + x + 1) = x(x-1)(x^2 + x + 1) - 5(x^2 + x + 1) = (x^2 - x - 5)(x^2 + x + 1) = 0$. So the real solution are $\dfrac{\mathbf{1 \pm \sqrt{21}}}{\mathbf{2}}$

50. Recall that $i^2 = -1$. Find the value of this complex number

$$\frac{1+i}{1} \cdot \frac{3+i}{2} \cdot \frac{7+i}{5} \cdot \frac{13+i}{10} \cdot \frac{21+i}{17} \cdots \frac{871+i}{842}$$

Solution: Note the pattern in the small cases: $\dfrac{1+i}{1} \cdot \dfrac{3+i}{2} = 1 + 2i$, $\dfrac{1+i}{1} \cdot \dfrac{3+i}{2} \cdot \dfrac{7+i}{5} = 1 + 3i$, $1 + i1 \cdot \dfrac{3+i}{2} \cdot \dfrac{7+i}{5} \cdot \dfrac{13+i}{10} = 1 + 4i$, etc. Since $\dfrac{871+i}{842}$ is the 30th factor in this product, then $\mathbf{1 + 30i}$ is the answer.

51. For all positive integers n and m,

$$f(mn+1) = f(n)f(m+1) + f(m)f(n+1) \quad \text{and} \quad f(n) > 0$$

Find the value of $f(11)$. Let $m = 0$ and $n = 0$. Then $f(1) = 2f(0)f(1)$ so $f(0) = 1/2$. Let $m = n = 1$, then $f(2) = 2f(1)f(2)$, so $f(1) = 1/2$. Let $m = 1$ and $n = 0$. Then $1/2 = 1/4 + (1/2)f(2)$ so $f(2) = 1/2$. Let $n = 1$, so $f(m+1) = f(m)(1/2) + (1/2)f(m+1)$. Thus $f(m+1) = f(m)$ for all positive integers thus $\mathbf{f(11) = 1/2}$.

52. In the coordinate plane a laser beam is fired from the origin. After hitting a mirror at $(1,7)$, the beam passes through the point $(15,5)$. The mirror is given by the graph of $ay - bx = c$, where a, b, and c are positive integers with a, b, and c relatively prime. What is the value of $a + b + c$? Solution: The equation of the mirror is $4y - 3x = 25$. The lines $4y - 3x = 0$, $y = 7x$ and the line containing $(1,7)$ and $(15,5)$ form an isosceles triangle with vertex at $(1,7)$ and of course its base is parallel to the mirror. This can be checked with the formula for $\tan(A + B)$. So the answer is $\mathbf{4 + 3 + 25 = 32}$.

53. Find $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e^{k/n}$.

Solution: Recognize this limit to be $\int_{\mathbf{0}}^{\mathbf{1}} \mathbf{e^x \, dx = e - 1}$.

54. The line $y = mx$ cuts in half the area of the region bounded by $y = 4x - x^2$ and the x-axis. Find the value of $(4 - m)^3$.

Solution: The area of the region is 32/3 so put $\int_0^{4-m} 4x - x^2 - mx \, dx = (4 - m)^3/6 = 32/6$. Thus $\mathbf{(4 - m)^3 = 32}$.

55. Evaluate $\int_1^{64} \frac{1}{\sqrt{x} + \sqrt[3]{x}} \, dx$. $11 - 6\ln(3/2)$

Solution: Put $u = x^{1/6}$ which leads to $\int_1^2 \frac{6u^5}{u^3 + u^2} \, du = \mathbf{11 - 6\ln(3/2)}$.

56. Given that for fixed constants A and B

$$\int \sin(2x)\cos(3x) \, dx = A\sin(2x)\sin(3x) + B\cos(2x)\cos(3x) + C$$

find the value of $A + B$.

Solution: Differentiate the right hand side to obtain the system $3A - 2B = 1$ and $2A - 3B = 0$. The solution is $A = 3/5$ and $B = 2/5$. So $\mathbf{A + B = 1}$.

57. Find the value of $\displaystyle\int_0^\infty \frac{\ln x}{1 + x^2} \, dx$.

Solution: Make the substitution $x = 1/u$, $dx = -1/u^2 du$, so that $\int_1^\infty \frac{\ln x}{1 + x^2} \, dx = -\int_0^1 \frac{\ln u}{u^2 + 1} \, du$. So $\displaystyle\int_{\mathbf{0}}^\infty \frac{\ln \mathbf{x}}{\mathbf{1 + x^2}} \, \mathbf{dx = 0}$.